

CUP PRODUCTS OF LINE BUNDLES ON HOMOGENEOUS VARIETIES AND GENERALIZED PRV COMPONENTS OF MULTIPLICITY ONE

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ABSTRACT. Let $X = G/B$ and let L_1 and L_2 be two line bundles on X . Consider the cup product map

$$H^{d_1}(X, L_1) \otimes H^{d_2}(X, L_2) \xrightarrow{\cup} H^d(X, L),$$

where $L = L_1 \otimes L_2$ and $d = d_1 + d_2$. We answer two natural questions about the map above: When is it a nonzero homomorphism of representations of G ? Conversely, given generic irreducible representations V_1 and V_2 , which irreducible components of $V_1 \otimes V_2$ may appear in the right hand side of the equation above? For the first question we find a combinatorial condition expressed in terms of inversion sets of Weyl group elements. The answer to the second question is especially elegant - the representations V appearing in the right hand side of the equation above are exactly the generalized PRV components of $V_1 \otimes V_2$ of stable multiplicity one. Furthermore, the highest weights $(\lambda_1, \lambda_2, \lambda)$ corresponding to the representations (V_1, V_2, V) fill up the generic faces of the Littlewood-Richardson cone of G of codimension equal to the rank of G . In particular, we conclude that the corresponding Littlewood-Richardson coefficients equal one.

Keywords: Homogeneous variety, Littlewood-Richardson coefficient, Borel-Weil-Bott theorem, PRV component.

CONTENTS

1. Introduction	2
2. Notation and background results	6
3. Diagonal Bott-Samelson-Demazure-Hansen-Kumar Varieties	17
4. Proof of Theorem III	28
5. Proof of Theorem I and corollaries	32
6. Cohomological components and proof of Theorem II	38
7. Littlewood-Richardson cone and bounds on multiplicities	45
A. Appendix on invariants of $\text{Sym}^\cdot \mathfrak{g}$	52
References	60

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1. INTRODUCTION

1.1. Main problems. The main object of study of this paper is the cup product map

$$(1.1.1) \quad H^{d_1}(X, L_1) \otimes \cdots \otimes H^{d_k}(X, L_k) \xrightarrow{\cup} H^d(X, L),$$

where $X = G/B$; G is a semisimple algebraic group over an algebraically closed field of characteristic zero, B is a Borel subgroup of G ; L_1, \dots, L_k are arbitrary line bundles on X , $L = L_1 \otimes \cdots \otimes L_k$; d_1, \dots, d_k are non-negative integers, and $d = d_1 + \cdots + d_k$.

We assume that both sides of (1.1.1) are nonzero for otherwise the cup product map is the zero map. Without loss of generality we may also assume that the line bundles L_1, \dots, L_k , and L are G -equivariant; then both sides of (1.1.1) carry a natural G -module structure and the cup product map is G -equivariant. Furthermore by the Borel-Weil-Bott theorem there are irreducible representations $V_{\mu_1}, \dots, V_{\mu_k}$, and V_μ so that $H^{d_i}(X, L_i) = V_{\mu_i}^*$ for $i = 1, \dots, k$, and $H^d(X, L) = V_\mu^*$ as representations of G . The dual of (1.1.1) is thus a G -homomorphism

$$(1.1.2) \quad V_\mu \longrightarrow V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}.$$

Since $V_{\mu_1}, \dots, V_{\mu_k}$, and V_μ are irreducible representations, (1.1.1) is either surjective or zero; respectively, (1.1.2) is either injective or zero. This leads us naturally to the two main problems of this paper.

Problem I. When is (1.1.1) a surjection of nontrivial representations?

Problem II. For which $(k+1)$ -tuples $(V_{\mu_1}, \dots, V_{\mu_k}, V_\mu)$ of irreducible representations of G can V_μ be realized as a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ via (1.1.2) for appropriate line bundles L_1, \dots, L_k on X ?

We call an irreducible representation V_μ which can be embedded into $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ via (1.1.2) a *cohomological component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. Fixing $V_{\mu_1}, \dots, V_{\mu_k}$, a variation of Problem II is to determine the cohomological components of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$.

With the exception of some quite degenerate cases for Problem II, we provide a complete solution to both problems.

1.2. Solution of Problem I. Fix a maximal torus $T \subseteq B$. The G -equivariant line bundles on X are in one-to-one correspondence with the characters of T . For a character λ of T , we denote by L_λ the line bundle on X corresponding to the one dimensional representation of B on which T acts via $-\lambda$.

The affine action of the Weyl group \mathcal{W} of G on the lattice of T -characters Λ is defined as

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

where ρ , as usual, denotes the half-sum of the roots of B . A character $\lambda \in \Lambda$ is *regular* if there exists a (necessarily unique) element $w \in \mathcal{W}$ such that $w \cdot \lambda$ is a dominant character. Following Kostant, [K1, Definition 5.10], we define the *inversion set* Φ_w of $w \in \mathcal{W}$ as the set $\Phi_w = w^{-1}\Delta^- \cap \Delta^+$, where $\Delta^- = -\Delta^+$ is the set of negative roots of G .

Let $\lambda_1, \dots, \lambda_k \in \Lambda$ be the (regular) characters such that $L_i = L_{\lambda_i}$ for $1 \leq i \leq k$. Then $L = L_\lambda$, where $\lambda = \sum_{i=1}^k \lambda_i$. Assume that λ is also regular and denote by w_1, \dots, w_k , and w the Weyl group elements for which $w_i \cdot \lambda_i$ for $1 \leq i \leq k$ and $w \cdot \lambda$ are dominant. With this notation we prove the following criterion for surjectivity of (1.1.1).

Theorem I — For any semisimple G , if $H^d(X, L_\lambda) \neq 0$, then the cup product map (1.1.1) is surjective if and only if

$$(1.2.1) \quad \Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}.$$

Studying the structure of $(k+1)$ -tuples (w_1, \dots, w_k, w) satisfying (1.2.1) is an interesting combinatorial problem which we do not address here. For some open questions concerning (1.2.1) see the expository article [DR].

1.3. Solution of Problem II. We say that a component V_μ of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ has *stable multiplicity one* if the multiplicity of $V_{m\mu}$ in $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ is one for all $m \gg 0$. We say that V_μ is a *generalized PRV component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist w_1, \dots, w_k , and $w \in \mathcal{W}$ such that $w^{-1}\mu = w_1^{-1}\mu_1 + \cdots + w_k^{-1}\mu_k$. (See §2.3 and §6.1 for further discussion of these conditions.)

Theorem II —

- (a) Let V_μ be a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. Then V_μ is a generalized PRV component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ of stable multiplicity one.
- (b) Conversely, assume that V_μ is a generalized PRV component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ of stable multiplicity one. If, in addition, one of the following holds:
 - (i) at least one of μ_1, \dots, μ_k or μ is strictly dominant,
 - (ii) G is a simple classical group or a product of simple classical groups,
then V_μ is a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$.

It is unfortunate that in part (b) above we require condition (i) or (ii). Indeed, we believe that we do not need these conditions but we impose them due to our inability to overcome a combinatorial problem.

1.4. Representation-theoretic implications of Theorem II. The representation-theoretic significance of Theorem II is twofold: it provides both a geometric construction of special components of a tensor product via the Bott theorem and a new way of generalizing the classical PRV component.

The Borel-Weil-Bott theorem provides a geometric realization of every irreducible representation of G as the cohomology (in any degree) of an appropriate line bundle on X . In particular, every irreducible representation equals the space of global sections of a unique line bundle on X . In this sense the Borel-Weil theorem (the statement about cohomology in degree zero) suffices since the Bott theorem (the statement about higher cohomology) yields the same representations. However, in addition to being representations, the cohomology groups carry a ring structure induced from the cup product. Theorem II employs this structure to give a geometric realization of certain components of a tensor product of

representations. As far as we know this is the first use of the Bott theorem for a geometric construction of representations in the case when G is a semisimple algebraic group over a field of characteristic zero.

We are borrowing the term "generalized PRV component" from the case when $k = 2$. In [PRV] Parthasarathy, Ranga Rao, and Varadarajan established that if μ is in the \mathcal{W} -orbit of $\mu_1 + w_0\mu_2$ (where w_0 denotes the longest element of \mathcal{W}), then V_μ is a component of $V_{\mu_1} \otimes V_{\mu_2}$. Moreover, they proved that V_μ has multiplicity one in, and is the smallest component of, $V_{\mu_1} \otimes V_{\mu_2}$. It is true more generally that if μ is in the \mathcal{W} -orbit of $\mu_1 + v\mu_2$ (where v is now an arbitrary element of \mathcal{W}) then V_μ is again a component of $V_{\mu_1} \otimes V_{\mu_2}$; this was established independently by Kumar [Ku2] and Mathieu [M].

Unlike the original PRV component, a generalized PRV component V_μ may have multiplicity greater than one in $V_{\mu_1} \otimes V_{\mu_2}$. However, by Theorem II, every cohomological component is a generalized PRV component of stable multiplicity one. The cohomological components also retain an aspect of the minimality of the original PRV component: every cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is extreme among all components of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ (see Proposition 7.1.1). These properties of cohomological components suggest that they may be viewed as the "true" analog of the original PRV component.

The following examples illustrate Theorem II and Proposition 7.1.1 when $k = 2$.

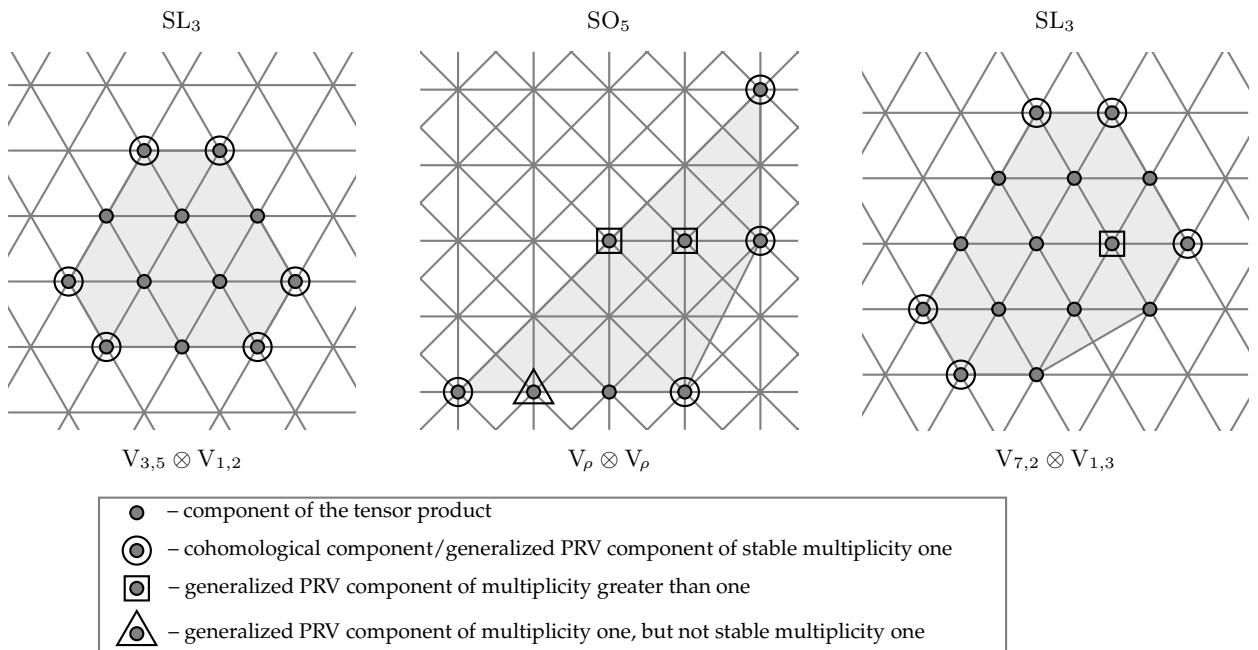


Figure 1

1.5. The combinatorics of the Littlewood-Richardson cone. The *Littlewood-Richardson cone* $\mathcal{C}(k)$ is defined as the rational cone generated by $(\mu_1, \dots, \mu_k, \mu)$ such that V_μ is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. Let n denote the rank of G . Combining Theorem I with results of Ressayre we obtain:

Theorem IV —

- (a) Assume that $(w_1, \dots, w_k, w) \in \mathcal{W}^{k+1}$ satisfies the conditions $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ and $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$. Then the integral weights $(\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k)$ such that V_μ is a cohomological component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$ with respect to (w_1, \dots, w_k, w) are exactly the lattice points of a codimension n face of $\mathcal{C}(k)$.
- (b) Conversely, given a codimension n face F of $\mathcal{C}(k)$ which intersects the locus of strictly dominant weights, there exists $(w_1, \dots, w_k, w) \in \mathcal{W}^{k+1}$ such that for every lattice point $(\mu_1, \dots, \mu_k, \mu)$ on F , V_μ is a cohomological component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$ with respect to (w_1, \dots, w_k, w) .

We also establish multiplicity bounds for the Littlewood-Richardson coefficients. Let p_{Δ^+} be the Kostant partition function and, for any integer $r \geq 1$, let $p_{r\Delta^+}$ be the partition function whose generating function is the r -th power of the generating function for p_{Δ^+} . For any w_1, \dots, w_k , and w such that (1.2.1) holds we prove the inequality (see (7.3.1))

$$(1.5.1) \quad \text{mult}(V_\mu, V_{\mu_1} \otimes \dots \otimes V_{\mu_k}) \leq p_{(k-1)\Delta^+} \left(\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu \right),$$

where $\text{mult}(V_\mu, V_{\mu_1} \otimes \dots \otimes V_{\mu_k})$ denotes the multiplicity of V_μ in $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$. Each of these bounds has the same asymptotic order of growth along rays as the multiplicity function. Applied to faces of $\mathcal{C}(2)$, (1.5.1) yields a strengthening of the statement of Theorem IV(b). Namely, if F is a face of $\mathcal{C}(2)$ of codimension $n-1$ which intersects the locus of strictly dominant weights, then for every triple (μ_1, μ_2, μ) of integral weights in F , the multiplicity of V_μ in $V_{\mu_1} \otimes V_{\mu_2}$ is at most one. After scaling (i.e., modulo the saturation problem) V_μ has multiplicity exactly one. Even this statement can be strengthened, see Corollary 7.3.5.

1.6. Other results. In conclusion we mention several other results which may be of independent interest.

The cup product and Schubert calculus. Recall that a basis for the cohomology ring $H^*(X, \mathbf{Z})$ of $X = G/B$ is given by the classes of the Schubert cycles $\{[X_w]\}_{w \in \mathcal{W}}$ indexed by the elements of the Weyl group \mathcal{W} . The dual basis $\{[\Omega_w]\}_{w \in \mathcal{W}}$, is given by $\Omega_w := X_{w_0 w}$. With the notation of §1.2 we prove the following:

Theorem III — For any semisimple algebraic group G ,

- (a) if $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$ then the cup product map (1.1.1) is surjective;
- (b) if $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 0$ then the cup product map (1.1.1) is zero.

We use Theorem III as stated above and a variation of its proof to prove Theorem I. If G is a product of simple classical groups, then one can show that condition (1.2.1) implies that $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$. We do not know if condition (1.2.1) implies that the intersection number is one for the exceptional groups of rank greater than two.

Diagonal Bott-Samelson-Demazure-Hansen varieties. We construct a class of varieties which generalize the Bott-Samelson-Demazure-Hansen varieties. These varieties may find applications beyond this paper. In particular we express them as resolutions of singularities of the total space of intersections of translates of Schubert varieties, cf. Theorem 3.7.4. Other notable results related to this construction include Lemma 3.8.1 which controls the multiplicity of cohomological components, and Theorem 3.9.1 which provides a new proof of the necessity of the inequalities determining the Littlewood-Richardson cone.

On invariants of $\text{Sym}^r(\mathfrak{g})$. The proof of Theorem II(b) requires the solution of a combinatorial problem which we reduce to a problem of finding a parabolic subalgebra of \mathfrak{g} with particular properties. The proof of the existence of such a subalgebra uses methods very different from the rest of the paper and is provided in a self-contained Appendix. The spirit of the result, Theorem A.1, is very classical. Notably, this is the only place in the paper where a case-by-case analysis is carried out. This is not an accident – Theorem A.1 does not hold when \mathfrak{g} is an exceptional Lie algebra of rank greater than two.

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2. NOTATION AND BACKGROUND RESULTS

2.1. Notation and conventions. The ground field is algebraically closed of characteristic zero. Throughout the paper we fix a semisimple connected algebraic group G , a Borel subgroup $B \subset G$, and a maximal torus $T \subset B$. All parabolic subgroups we consider contain T . The Lie algebras of algebraic groups are denoted by fraktur letters, e.g. $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$, etc. We use the term “ G -module” instead of “representation of G ” to avoid differentiating between representations of algebraic groups and modules over the respective Lie algebras; likewise, since T is fixed, we use the term “weight” both for characters of T and weights of \mathfrak{t} ; in particular we only consider integral weights of \mathfrak{t} .

The point $wB/B \in X_w \subseteq X = G/B$, where $w \in \mathcal{W}$ and X_w is the corresponding Schubert variety, is denoted by w for short. If $M = G/P$ for some parabolic P we similarly use w to indicate the point $wP/P \in M$.

If Λ is the lattice of weights of T we denote the group ring of Λ by $\mathbf{Z}[\Lambda]$, i.e.

$$\mathbf{Z}[\Lambda] = \left\{ \sum_{i=1}^k c_i e^{\lambda_i} \mid c_i \in \mathbf{Z}, \lambda_i \in \Lambda \right\}.$$

For a T -module \mathcal{M} , the *formal character* of \mathcal{M} is

$$\text{Ch } \mathcal{M} = \sum_{\lambda \in \Lambda} \dim \mathcal{M}^\lambda e^\lambda \in \mathbf{Z}[\Lambda],$$

where $\mathcal{M}^\lambda = \{x \in \mathcal{M} \mid t \cdot x = \lambda(t)x \text{ for every } t \in \mathfrak{t}\}$. All formal characters discussed in this paper are contained in $\mathbf{Z}[\Delta]$. For a subset $\Phi \subseteq \Delta$, the formal character of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha$ is

denoted by $\langle \Phi \rangle$, i.e.

$$\langle \Phi \rangle = \sum_{\alpha \in \Phi} e^\alpha.$$

If w is an element of the Weyl group \mathcal{W} , then $\ell(w)$ means the length of any minimal expression giving w as a product of simple reflections. If \underline{v} is a word in the simple reflections, then $\ell(\underline{v})$ is the number of reflections in the word. Note that, if \underline{v} is a word in simple reflections, and $v \in \mathcal{W}$ is the corresponding element of the Weyl group, then $\ell(\underline{v}) = \ell(v)$ if and only if \underline{v} is a reduced word. If $\underline{v} = s_{i_1} \dots s_{i_m}$ is a non-empty word, we denote by \underline{v}_R the word $s_{i_1} \dots s_{i_{m-1}}$ obtained from \underline{v} by dropping the rightmost reflection in \underline{v} . If $\underline{\mathbf{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ is a sequence of words then we set $\ell(\underline{\mathbf{v}}) = \sum_{i=1}^k \ell(\underline{v}_i)$.

The following notation is used consistently throughout the paper.

$\sqcup_{i=1}^k$	- disjoint union
$\kappa(\cdot, \cdot)$	- the Killing form of G
Λ, Λ^+	- weight lattice and cone of dominant weights
V_μ	- irreducible G -module of highest weight μ
$\text{mult}(V_\mu, V)$	- the multiplicity of V_μ in V
$\mathcal{C}(k)$	- the Littlewood-Richardson cone in Λ^{k+1} , see §2.7
$\mathcal{P}(\mu_1, \dots, \mu_k)$	- the Littlewood-Richardson polytope, see §2.7
$\{\alpha_1, \dots, \alpha_n\}$	- base of simple roots of B
\mathcal{W}	- Weyl group of \mathfrak{g}
$w \cdot \lambda$	- $w(\lambda + \rho) - \rho$, the result of the affine action of $w \in \mathcal{W}$ on $\lambda \in \Lambda$
s_i	- simple reflection along α_i
P_{α_i}	- the minimal parabolic subgroup of G associated to α_i
P_I	- the minimal parabolic subgroup of G associated to a set I of simple roots
\mathcal{W}_P	- the Weyl group of a parabolic subgroup $P \subseteq G$
$\text{span}_{\mathbb{Z}_{\geq 0}} \Phi$	- the set of non-negative integer combinations of elements of $\Phi \subseteq \Delta$
\underline{u} or \underline{v}	- a word $s_{i_1} \dots s_{i_m}$ in the simple reflections of the Weyl group
u, v	- the element of \mathcal{W} corresponding to \underline{u} or \underline{v}
\underline{v}_R	- the word obtained by dropping the rightmost reflection of \underline{v}
$\underline{\mathbf{u}}$ or $\underline{\mathbf{v}}$	- a sequence $(\underline{u}_1, \dots, \underline{u}_k)$ or $(\underline{v}_1, \dots, \underline{v}_k)$ of words
Φ_w	- $w^{-1}\Delta^- \cap \Delta^+$, the inversion set of $w \in \mathcal{W}$
$\langle \Phi \rangle$	- $\sum_{\alpha \in \Phi} e^\alpha$, the formal character of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha$, where $\Phi \subset \Delta$
$\ell(w)$	- the length of $w \in \mathcal{W}$
L_λ	- the line bundle on X corresponding to B -module on which T acts via $-\lambda$
N	- the dimension of X
π_i	- the projection $\pi_i: X \longrightarrow G/P_{\alpha_i}$ (a \mathbf{P}^1 -fibration)

2.2. Inversion sets. Let Δ^+ be the set of positive roots of \mathfrak{g} (with respect to B). Following Kostant [K1, Definition 5.10], for any element w of the Weyl group \mathcal{W} we define Φ_w , the *inversion set* of w , to be the set of positive roots sent to negative roots by w , i.e.,

$$(2.2.1) \quad \Phi_w := w^{-1}\Delta^- \cap \Delta^+.$$

For a subset Φ of Δ^+ , we set $\Phi^c := \Delta^+ \setminus \Phi$. We will need the following formulas, which follow easily from the definition:

$$(2.2.2) \quad \Phi_{w_0 w} = \Phi_w^c;$$

$$(2.2.3) \quad w^{-1} \Delta^+ = \Phi_w^c \sqcup -\Phi_w;$$

$$(2.2.4) \quad w^{-1} \cdot 0 = w^{-1} \rho - \rho = - \sum_{\alpha \in \Phi_w} \alpha.$$

2.3. Generalized PRV components. For fixed dominant weights μ_1, μ_2 , and μ it is clear that the two conditions

- (a) there exist w_1, w_2 , and w in \mathcal{W} such that $w^{-1}\mu = w_1^{-1}\mu_1 + w_2^{-1}\mu_2$,
- (b) there exists v in \mathcal{W} such that μ is in the \mathcal{W} -orbit of $\mu_1 + v\mu_2$

are equivalent. If these conditions are satisfied we call V_μ a *generalized PRV component* of $V_{\mu_1} \otimes V_{\mu_2}$.

As is suggested by the name, but is far from obvious from the definition, every generalized PRV component of $V_{\mu_1} \otimes V_{\mu_2}$ is in fact a component of the tensor product $V_{\mu_1} \otimes V_{\mu_2}$ of G -modules. This was first proved when $v = w_0$ (i.e., when μ is in the \mathcal{W} -orbit of $\mu_1 + w_0\mu_2$) in [PRV]. In the literature this component is referred to simply as the *PRV component*. The general case, that V_μ is a component of $V_{\mu_1} \otimes V_{\mu_2}$ for an arbitrary v , became known as the PRV conjecture, and was established independently by Kumar [Ku2] and Mathieu [M].

In the present paper we extend the notion of generalized PRV component to components of the tensor product of k irreducible G -modules for $k \geq 2$. We call V_μ a *generalized PRV component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist w_1, \dots, w_k , and w in \mathcal{W} such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$. A straightforward induction from the case $k = 2$ implies that every generalized PRV component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is a component of the tensor product $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ of G -modules. We record the special case when $\mu = 0$ for use in the proof of Theorem I.

Lemma (2.3.1) — For any dominant weights μ_1, \dots, μ_k , and Weyl group elements w_1, \dots, w_k , if $\sum_{i=1}^k w_i^{-1}\mu_i = 0$ then $(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G \neq 0$.

2.4. Borel-Weil-Bott theorem. Suppose that λ is a regular weight, so there is a unique $w \in \mathcal{W}$ with $w \cdot \lambda \in \Lambda^+$. The Borel-Weil-Bott theorem identifies the cohomology of the line bundle L_λ on X as G -modules:

$$H^d(X, L_\lambda) = \begin{cases} V_{w \cdot \lambda}^* & \text{if } d = \ell(w) \\ 0 & \text{otherwise.} \end{cases}$$

If λ is not a regular weight then the cohomology of L_λ is zero in all degrees.

2.5. Serre Duality on X . For any weight λ set $S(\lambda) = -\lambda - 2\rho$. Since the canonical bundle K_X of X is equal to $L_{-2\rho}$ we see that $L_{S(\lambda)} = K_X \otimes L_\lambda^*$. In other words, S is the function

that for each weight λ returns the weight $S(\lambda)$ of the line bundle Serre dual to L_λ ; the map S is clearly an involution. Let w be any element of the Weyl group and λ any weight. A straightforward computation shows that S commutes with the affine action of the Weyl group, i.e. that $w \cdot S(\lambda) = S(w \cdot \lambda)$.

Lemma (2.5.1) — If λ is a regular weight and w the unique element of the Weyl group with $w \cdot \lambda \in \Lambda^+$ then $(w_0 w) \cdot S(\lambda) \in \Lambda^+$.

Proof. If μ is a dominant weight then $V_\mu^* = V_{-w_0\mu}$. Therefore if $w \cdot \lambda = \mu \in \Lambda^+$ then

$$(2.5.2) \quad (w_0 w) \cdot S(\lambda) = w_0 \cdot S(w \cdot \lambda) = w_0 \cdot S(\mu) = -w_0 \mu \in \Lambda^+. \quad \square$$

Since $\ell(w_0 w) = N - \ell(w)$, the calculation above fits in neatly with the Borel-Weil-Bott theorem and Serre duality. If λ is a regular weight and w an element of the Weyl group with $w \cdot \lambda = \mu \in \Lambda^+$ then we have

$$V_\mu \xrightarrow{\text{BWB}} (H^{\ell(w)}(X, L_\lambda))^* \xrightarrow{\text{Serre}} H^{N-\ell(w)}(X, K_X \otimes L_\lambda^*) \xrightarrow[+ (2.5.2)]{2.5} H^{\ell(w_0 w)}(X, L_{S(\lambda)}) \xrightarrow{\text{BWB}} V_{-w_0 \mu}^*.$$

2.6. Schubert varieties. For an element $w \in \mathcal{W}$ of the Weyl group the *Schubert variety* X_w is defined by

$$X_w := \overline{BwB/B} \subseteq G/B = X.$$

Recall that the classes of the Schubert cycles $\{[X_w]\}_{w \in \mathcal{W}}$ give a basis for the cohomology ring $H^*(X, \mathbf{Z})$ of X . Each $[X_w]$ is a cycle of complex dimension $\ell(w)$. The dual Schubert cycles $\{[\Omega_w]\}_{w \in \mathcal{W}}$, given by $\Omega_w := X_{w_0 w}$, also form a basis. Each $[\Omega_w]$ is a cycle of complex codimension $\ell(w)$. The work of Demazure [De1], Kempf [Ke], Ramanathan [R], and Seshadri [S] shows that each Schubert variety X_w is normal with rational singularities.

Remark. If w_1, \dots, w_k , and $w \in \mathcal{W}$ are such that $\ell(w) = \sum \ell(w_i)$, then the intersection $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w]$ is a number. The number is the coefficient of $[\Omega_w]$ when writing the product $\bigcap_{i=1}^k [\Omega_{w_i}]$ in terms of the basis $\{[\Omega_v]\}_{v \in \mathcal{W}}$.

To reduce notation we use w to also refer to the point $wB/B \in X_w \subseteq X$. In particular for the identity $e \in \mathcal{W}$, $X_e = \{e\}$. Note that $e \in X$ is also the image of 1_G under the projection from G onto X .

Bruhat order. The *Bruhat order* on the Weyl group \mathcal{W} is the partial order given by the relation $v \leqslant w$ if and only if $X_v \subseteq X_w$. The minimum element in this order is e and the maximum element is w_0 , corresponding to the subvarieties $X_e = \{e\}$ and $X_{w_0} = X$ respectively.

The following result will be used several times throughout the paper.

Lemma (2.6.1) — Suppose that w_1, \dots, w_k are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$. Then $\bigcap_{i=1}^k [\Omega_{w_i}] \neq 0$.

Proof. Each class $[\Omega_{w_i}]$ is represented by any translation of the cycle Ω_{w_i} , so to understand $\cap_{i=1}^k [\Omega_{w_i}]$ we can study the intersection of schemes

$$(2.6.2) \quad \bigcap_{i=1}^k (w_0 w_i)^{-1} \Omega_{w_i}.$$

Each of the schemes $(w_0 w_i)^{-1} \Omega_{w_i}$ passes through $e \in X$. The tangent space to $(w_0 w)^{-1} \Omega_w$ at e is

$$\text{Lie}((w_0 w)^{-1} B(w_0 w)) / \text{Lie}(B) = \bigoplus_{\alpha \in -\Phi_{w_0 w}} \mathfrak{g}^\alpha \stackrel{(2.2.2)}{=} \bigoplus_{\alpha \in -\Phi_w^c} \mathfrak{g}^\alpha \subseteq \mathfrak{b}^- = T_e X,$$

where we have identified $T_e X$ with \mathfrak{b}^- via the projection $G \rightarrow X$. Noting that

$$\bigcap_{i=1}^k \Phi_w^c = \left(\bigcup_{i=1}^k \Phi_w \right)^c = (\Delta^+)^c = \emptyset,$$

we conclude that the intersection of the tangent spaces of the varieties $(w_0 w_i)^{-1} \Omega_{w_i}$ at $e \in X$ is 0. Hence the intersection (2.6.2) is transverse at the identity. By Kleiman's transversality theorem [Kl, Corollary 4(ii)], small translations of each of the varieties $(w_0 w_i)^{-1} \Omega_{w_i}$ will intersect properly and compute the intersection number. Small translations of varieties cannot remove transverse points of intersection and thus $\cap_{i=1}^k [\Omega_{w_i}] \neq 0$. \square

2.7. The Littlewood-Richardson Cone. For any $k \geq 1$, let $\mathcal{C}(k)$ be the *Littlewood-Richardson cone*, i.e., the rational cone generated by the tuples $(\mu_1, \dots, \mu_k, \mu)$ of dominant weights such that V_μ is a component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$. It is known that $\mathcal{C}(k)$ is polyhedral. A (nonminimal) set of equations describing $\mathcal{C}(k)$ is given by the conditions that $(\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k)$ if and only if, for every $w_1, \dots, w_k, w \in \mathcal{W}$ such that $\sum_{i=1}^k \ell(w_i) = \ell(w)$ and $\cap_{i=1}^k [\Omega_{w_i}] \cdot X_w \neq 0$, the weight $\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu$ belongs to $\text{span}_{\mathbb{Z}_{\geq 0}} \Delta^+$ (see [BeSj, Theorem 5.2.1]).

For any dominant weights μ_1, \dots, μ_k , let $\mathcal{P}(\mu_1, \dots, \mu_k)$ be the slice of $\mathcal{C}(k)$ obtained by fixing the first k coordinates to be (μ_1, \dots, μ_k) . We call $\mathcal{P}(\mu_1, \dots, \mu_k)$ the *Littlewood-Richardson polytope* of μ_1, \dots, μ_k .

Results of Ressayre. For any set I of simple roots, we define P_I to be the parabolic subgroup associated to I . For any parabolic $P \supseteq B$ we denote the Weyl group of P by \mathcal{W}_P .

For a set I of simple roots we wish to consider elements w_1, \dots, w_k , and w of \mathcal{W} satisfying the following conditions with respect to I :

$$(2.7.1) \quad \left\{ \begin{array}{l} (i) \quad \ell(w) = \sum_{i=1}^k \ell(w_i) \text{ and } \cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1. \\ (ii) \quad \text{Each } w_i \text{ is of minimal length in the coset } w_i \mathcal{W}_{P_I} \text{ and } w \text{ is of minimal length in the coset } w \mathcal{W}_{P_I}. \\ (iii) \quad \text{The weight } \sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 \text{ belongs to } \text{span}_{\mathbb{Z}_{\geq 0}} I. \end{array} \right.$$

The work of Ressayre gives an explicit description of some of the boundary components of $\mathcal{C}(k)$. The following is a translation of [Re, Theorem D] into our notation:

Theorem (2.7.2) —

- (a) Let I be a set of simple roots and w_1, \dots, w_k , and w elements of \mathcal{W} satisfying conditions (2.7.1) with respect to I . Then the set

$$\left\{ (\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k) \mid \sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu \in \text{span}_{\mathbb{Q}_{\geq 0}} I \right\}$$

is a face of codimension $(n - \#I)$ of $\mathcal{C}(k)$.

- (b) Any face of $\mathcal{C}(k)$ which intersects the locus of strictly dominant weights is of the type in part (a).

2.8. Symmetric and nonsymmetric forms. Most questions we consider, including Problem I and Problem II, can be stated in nonsymmetric and symmetric forms and it is frequently convenient to switch from one to the other. We illustrate this procedure by showing how to switch from the nonsymmetric to the symmetric form of Problem I.

In the nonsymmetric form we are given w_1, \dots, w_k , and w , such that $\ell(w) = \sum \ell(w_i)$, and $\lambda_1, \dots, \lambda_k$, and λ , such that $\lambda = \sum \lambda_i$, satisfying the additional conditions that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \dots, k$, and $w \cdot \lambda \in \Lambda^+$. This corresponds to the data of a cup product problem:

$$(2.8.1) \quad H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^{\ell(w)}(X, L_{\lambda}).$$

Set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \dots, k$, and $\mu = w \cdot \lambda$ to keep track of the modules which appear as cohomology groups. By the Borel-Weil-Bott theorem the map (2.8.1) corresponds to a G -equivariant map

$$V_{\mu_1}^* \otimes V_{\mu_2}^* \otimes \cdots \otimes V_{\mu_k}^* \longrightarrow V_{\mu}^*.$$

By Serre duality $H^{N-\ell(w)}(X, K_X \otimes L_{\lambda}^*) \neq 0$ and the cup product map

$$H^{\ell(w)}(X, L_{\lambda}) \otimes H^{N-\ell(w)}(X, K_X \otimes L_{\lambda}^*) \xrightarrow{\cup} H^N(X, K_X)$$

is a perfect pairing. Since $H^{\ell(w)}(X, L_{\lambda})$ is an irreducible G -module, the surjectivity of (2.8.1) is equivalent to the surjectivity of the cup product map

$$(2.8.2) \quad H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \otimes H^{N-\ell(w)}(X, K_X \otimes L_{\lambda}^*) \xrightarrow{\cup} H^N(X, K_X).$$

To get the symmetric form of this problem, we set $w_{k+1} = w_0 w$, $\lambda_{k+1} = S(\lambda) = -\lambda - 2\rho$, and $\mu_{k+1} = -w_0 \mu = w_{k+1} \cdot \lambda_{k+1}$. Then $L_{\lambda_{k+1}} = K_X \otimes L_{\lambda}^*$ by §2.5, $w_{k+1} \cdot \lambda_{k+1} \in \Lambda^+$ by Lemma 2.5.1, and $\ell(w_{k+1}) = N - \ell(w)$, so that (2.8.2) becomes

$$H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \otimes H^{\ell(w_{k+1})}(X, L_{\lambda_{k+1}}) \xrightarrow{\cup} H^N(X, K_X).$$

Since $\sum_{i=1}^{k+1} \lambda_i = \lambda + (-\lambda - 2\rho) = -2\rho$ and $L_{-2\rho} = K_X$, this is again a cup product problem of the type we consider, but now all weights $\lambda_1, \dots, \lambda_{k+1}$ and Weyl group elements w_1, \dots, w_{k+1} play equal roles.

By (2.2.2) $\Phi_{w_{k+1}} = \Phi_w^c$ and therefore the condition that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ is equivalent to the condition $\Delta^+ = \sqcup_{i=1}^{k+1} \Phi_{w_i}$. Since $[\Omega_{w_{k+1}}] = [X_{w_0 w_{k+1}}] = [X_w]$, the intersection numbers $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w]$ and $\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$ are the same. Finally, the multiplicity of V_μ in $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is the same as the multiplicity of the trivial module in $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k} \otimes V_{\mu_{k+1}}$ because $V_{\mu_{k+1}} = V_{-w_0 \mu} = V_\mu^*$.

To go from the symmetric form to the nonsymmetric form we simply reverse the above procedure, although of course we are free to desymmetrize with respect to any of the indices $i = 1, \dots, k+1$, and not just the last one.

For convenience we list below the symmetric and nonsymmetric forms of some formulas and expressions we are interested in.

Nonsymmetric	Symmetric
$\bigotimes_{i=1}^k H^{\ell(w_i)}(X, L_{\lambda_i}) \longrightarrow H^{\ell(w)}(X, L_\lambda)$	$\bigotimes_{i=1}^{k+1} H^{\ell(w_i)}(X, L_{\lambda_i}) \longrightarrow H^N(X, K_X)$
$\sum_{i=1}^k \ell(w_i) = \ell(w)$	$\sum_{i=1}^{k+1} \ell(w_i) = N$
$\sum_{i=1}^k \lambda_i = \lambda$	$\sum_{i=1}^{k+1} \lambda_i = -2\rho$
$\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu$	$\sum_{i=1}^{k+1} w_i^{-1} \mu_i$
$\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0$	$\sum_{i=1}^{k+1} w_i^{-1} \cdot 0 + 2\rho$
$\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$	$\Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{w_i}$
$\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w]$	$\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$

Since k is an arbitrary positive integer, after switching to the symmetric form we often use k in place of $k+1$ to reduce notation.

2.9. Demazure reflections. Suppose that W and M are varieties and $\pi: W \longrightarrow M$ is a P^1 -fibration, i.e., a smooth morphism with fibres isomorphic to P^1 . Let L be a line bundle

on W and b be the degree of L on the fibres of π . Demazure [De2, Theorem 1] proves the following isomorphism of vector bundles on M :

$$(2.9.1) \quad R^i\pi_*L \cong R^{1-i}\pi_*\left(L \otimes \omega_\pi^{b+1}\right) \quad \text{for } i = 0, 1.$$

where ω_π is the relative cotangent bundle of π . The line bundle $L \otimes \omega_\pi^{b+1}$ is called the *Demazure reflection of L with respect to π* .

Note that there is at most one value of i for which the resulting vector bundles are nonzero: $i = 0$ if $b \geq 0$, $i = 1$ if $b \leq -2$, and neither if $b = -1$. Equation (2.9.1) and the corresponding Leray spectral sequence give the isomorphisms

$$H^j(W, L) \cong \begin{cases} H^{j+1}(W, L \otimes \omega_\pi^{b+1}) & \text{if } b \geq 0 \\ H^{j-1}(W, L \otimes \omega_\pi^{b+1}) & \text{if } b \leq -2 \end{cases} \quad \text{for all } j.$$

Link between Demazure reflections and the affine action. Let α_i be any simple root, P_{α_i} the parabolic associated to α_i , and $\pi_i: X \rightarrow M_i := G/P_{\alpha_i}$ the corresponding P^1 -fibration. The relative cotangent bundle ω_{π_i} of π_i is the line bundle $L_{-\alpha_i}$. Given any $\lambda \in \Lambda$, the degree of the line bundle L_λ on the fibres of π_i is $\lambda(\alpha_i^\vee)$ where α_i^\vee is the coroot corresponding to α_i . We thus obtain that the Demazure reflection of L_λ with respect to the fibration π_i is the line bundle:

$$L_\lambda \otimes L_{-\alpha_i}^{\lambda(\alpha_i^\vee)+1} = L_{\lambda-(\lambda(\alpha_i^\vee)+1)\alpha_i} = L_{s_i\lambda-\alpha_i} = L_{s_i\cdot\lambda}$$

where s_i is the simple reflection corresponding to α_i . The combinatorics of performing Demazure reflections with respect to the various P^1 -fibrations of X is therefore kept track of by the affine action of the Weyl group on Λ . In particular, if $v = s_{i_1} \cdots s_{i_m} \in \mathcal{W}$ and $\lambda \in \Lambda$ the result of applying the Demazure reflections with respect to the fibrations $\pi_{i_m}, \pi_{i_{m-1}}, \dots, \pi_{i_1}$ in that order to L_λ is $L_{v\cdot\lambda}$.

Demazure reflections and base change. Given any morphism $h: Y_2 \rightarrow M$ we can form the fibre product diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & W \\ \downarrow \pi_1 & \square & \downarrow \pi \\ Y_2 & \xrightarrow{h} & M \end{array} .$$

If π is a P^1 -fibration then so is π_1 , and $\omega_{\pi_1} = f^*\omega_\pi$. Therefore, for any line bundle L on V , we have

$$f^*(L \otimes \omega_\pi^{b+1}) = (f^*L) \otimes \omega_{\pi_1}^{b+1}$$

where b is the degree of L on the fibres of π . The degree of f^*L on π_1 is also b and therefore the formula above shows that the pullback of the Demazure reflection of L with respect to π is the Demazure reflection of the pullback of L with respect to π_1 . Furthermore, by the theorem on cohomology and base change, the natural morphisms

$$R^i\pi_{1*}(f^*L) \xleftarrow{\sim} h^*(R^i\pi_*L) \quad \text{and} \quad R^{1-i}\pi_{1*}((f^*L) \otimes \omega_{\pi_1}^{b+1}) \xleftarrow{\sim} h^*(R^{1-i}\pi_*(L \otimes \omega_\pi^{b+1}))$$

are isomorphisms for $i = 0, 1$.

2.10. E_2 -terms and computation of maps on cohomology. Suppose that we have a commutative diagram of varieties

$$\begin{array}{ccc} W' & \xhookrightarrow{\gamma} & W \\ \downarrow \pi' & & \downarrow \pi \\ M' & \xhookrightarrow{\quad} & M \end{array}$$

where the vertical maps are proper and the horizontal maps are closed immersions. Suppose further that we have coherent sheaves \mathcal{F} on W and \mathcal{F}' on W' , and a map $\varphi: \gamma^*\mathcal{F} \rightarrow \mathcal{F}'$ of sheaves on W' . The map φ induces maps $\varphi_d: H^d(W, \mathcal{F}) \rightarrow H^d(W', \mathcal{F}')$ on cohomology and maps $\varphi_{d,k}: H^{d-k}(M, R_{\pi*}^k \mathcal{F}) \rightarrow H^{d-k}(M', R_{\pi'*}^k \mathcal{F}')$ on the E_2 -terms of the Leray spectral sequences for \mathcal{F} and \mathcal{F}' with respect to π and π' . Assume that both spectral sequences degenerate at the E_2 -term. In §5.4 we will need to know when we can compute φ_d by knowing the maps $\varphi_{d,k}$.

By the definition of convergence of a spectral sequence there are increasing filtrations

$$0 = U_{-1} \subseteq U_0 \subseteq \cdots \subseteq U_d = H^d(W, \mathcal{F}) \text{ and } 0 = U'_{-1} \subseteq U'_0 \subseteq \cdots \subseteq U'_d = H^d(W', \mathcal{F}')$$

such that $U_k/U_{k-1} = H^{d-k}(M, R_{\pi*}^k \mathcal{F})$ and $U'_k/U'_{k-1} = H^{d-k}(M', R_{\pi'*}^k \mathcal{F}')$ for $k = 0, \dots, d$. Since the map φ_d on the cohomology groups is compatible with the filtrations (in the sense that $\varphi_d(U_k) \subseteq U'_k$ for $k = -1, \dots, d$), φ_d induces maps between the associated graded pieces of the filtrations; these maps are exactly the maps $\varphi_{d,k}$.

We will need to know that φ_d can be computed from the maps $\varphi_{d,k}$ in an elementary case. Suppose there is a unique k such that U_k/U_{k-1} is nonzero (and so $U_k/U_{k-1} = H^d(W, \mathcal{F})$), and a unique k' such that $U'_{k'}/U'_{k'-1}$ is nonzero (and so $U'_{k'}/U'_{k'-1} = H^d(W', \mathcal{F}')$). Then we can compute φ_d from the maps $\varphi_{d,k}$ if and only if $k = k'$; if this occurs then $\varphi_d = \varphi_{d,k}$.

In order to show that we must check the condition $k = k'$ above, i.e., that the map on E_2 -terms does not always determine the map φ_d , we give the following example of a nonzero map between cohomology groups of sheaves where the induced map on E_2 -terms is zero. This example is also a cup product map.

Example (2.10.1) — Let $W = \mathbf{P}^m \times \mathbf{P}^m$ for some $m \geq 1$, $\mathcal{F} = \mathcal{O}_{\mathbf{P}^m}(1) \boxtimes \mathcal{O}_{\mathbf{P}^m}(-r)$ with $r \geq m+2$, and let $\mathcal{G} = \mathcal{O}_\Delta(1-r)$ be the restriction of \mathcal{F} to the diagonal of W . We have $H^m(W, \mathcal{F}) = H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1)) \otimes H^m(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(-r))$ and $H^m(W, \mathcal{G}) = H^m(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1-r))$. The natural restriction map $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ induces the cup product map

$$\varphi_m: H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1)) \otimes H^m(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(-r)) \xrightarrow{\cup} H^m(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1-r))$$

which is a surjective map of nonzero groups.

If $\pi: W \rightarrow M = \mathbf{P}^m$ is the projection onto the first factor then both of the Leray spectral sequences degenerate at the E_2 term with only one nonzero entry in each sequence. We have $H^m(W, \mathcal{F}) = H^0(M, R_{\pi*}^m \mathcal{F})$ (i.e., $k = m$) and $H^m(W, \mathcal{G}) = H^m(M, \pi_* \mathcal{G})$ (i.e., $k' = 0$). The maps $\varphi_{m,k}$ on the E_2 -terms are clearly zero, even though φ_m is nonzero.

2.11. Bott-Samelson-Demazure-Hansen Varieties. Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a word, not necessarily reduced, of simple reflections. Associated to \underline{v} is a variety $Z_{\underline{v}}$, a left action of B on $Z_{\underline{v}}$, and a B -equivariant map $f_{\underline{v}}: Z_{\underline{v}} \rightarrow X$. If \underline{v} is nonempty there is also a B -equivariant map $\pi_{\underline{v}}: Z_{\underline{v}} \rightarrow Z_{\underline{v}_R}$ expressing $Z_{\underline{v}}$ as a P^1 -bundle over $Z_{\underline{v}_R}$ together with a B -equivariant $\sigma_{\underline{v}}: Z_{\underline{v}_R} \rightarrow Z_{\underline{v}}$ section such that $f_{\underline{v}_R} = f_{\underline{v}} \circ \sigma_{\underline{v}}$.

These varieties were originally constructed by Demazure [De1] and Hansen [Ha] following an analogous construction by Bott and Samelson [BS] in the compact case. In this subsection we recall their construction and several related facts. We give two different descriptions of the construction; both will be used in the constructions in section 3.

Recursive Construction. Recall that e is unique point of X fixed by B . If the word \underline{v} is empty we define $Z_{\underline{v}}$ to be e , the map $f_{\underline{v}}$ to be the inclusion $e \hookrightarrow X$, and the B -action on $Z_{\underline{v}}$ to be trivial.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty, let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ be the word obtained by dropping the rightmost reflection of \underline{v} . By induction we have already constructed $Z_{\underline{u}}$ and the map $f_{\underline{u}}: Z_{\underline{u}} \rightarrow X$. Set $h = \pi_{i_m} \circ f_{\underline{u}}$, where π_{i_m} is the G -equivariant projection (and P^1 -fibration) $X \rightarrow M_{i_m} = G/P_{\alpha_{i_m}}$. We then define $Z_{\underline{v}}$ to be the fibre product $Z_{\underline{u}} \times_{M_{i_m}} X$, and $f_{\underline{v}}$ and $\pi_{\underline{v}}$ to be the maps from the fibre product to X and to $Z_{\underline{u}}$ respectively. Since $h = \pi_{i_m} \circ f_{\underline{u}}$ by the universal property of the fibre product there exists a unique map $\sigma_{\underline{v}}: Z_{\underline{u}} \rightarrow Z_{\underline{v}}$ such that $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}}$ and $\text{id}_{Z_{\underline{u}}} = \pi_{\underline{v}} \circ \sigma_{\underline{v}}$. These maps are summarized in the following diagram, where the square is a fibre product:

$$(2.11.1) \quad \begin{array}{ccc} Z_{\underline{v}} & \xrightarrow{f_{\underline{v}}} & X \\ \sigma_{\underline{v}} \swarrow \downarrow \pi_{\underline{v}} & \square & \downarrow \pi_{i_m} \\ Z_{\underline{u}} & \xrightarrow{h} & M_{i_m} \end{array} .$$

Since B acts on $Z_{\underline{u}}$ and on X , and the maps $f_{\underline{u}}$, π_{i_m} , and h are B -equivariant, by the universal property of the fibre product, the diagram (2.11.1) induces a B -action on $Z_{\underline{v}}$ such that $f_{\underline{v}}$ and $\sigma_{\underline{v}}$ are B -equivariant maps. Since each morphism $\sigma_{\underline{v}}$ is a P^1 -fibration it follows immediately that each $Z_{\underline{v}}$ is a smooth proper variety of dimension $\ell(\underline{v})$.

Direct Construction. For any word \underline{v} set

$$P_{\underline{v}} := \begin{cases} e & \text{if } \underline{v} \text{ is empty} \\ P_{\alpha_{i_1}} \times \cdots \times P_{\alpha_{i_m}} & \text{if } \underline{v} = s_{i_1} \cdots s_{i_m} \text{ is nonempty.} \end{cases}$$

If \underline{v} is empty we define $Z_{\underline{v}}$, $f_{\underline{v}}$, and the B -action as in the direct construction.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty then $Z_{\underline{v}}$ is the quotient of $P_{\underline{v}}$ by B^m , where an element (b_1, \dots, b_m) of B^m acts on the right on (p_1, \dots, p_m) by

$$(p_1, \dots, p_m) \cdot (b_1, \dots, b_m) = (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \dots, b_{m-1}^{-1} p_m b_m).$$

The left action of B on $P_{\underline{v}}$ given by

$$b \cdot (p_1, p_2, \dots, p_m) = (bp_1, p_2, \dots, p_m)$$

commutes with the right action of B^m and therefore descends to a left action of B on $Z_{\underline{v}}$. We denote the corresponding B -equivariant quotient map by $\psi_{\underline{v}}: P_{\underline{v}} \rightarrow Z_{\underline{v}}$.

The product map $P_{\underline{v}} \xrightarrow{\phi_{\underline{v}}} G$ given by $(p_1, \dots, p_m) \mapsto p_1 \cdots p_m$ is equivariant for the left B -action described above and left multiplication of G by B . Under the homomorphism of groups $B^m \rightarrow B$ given by the projection $(b_1, \dots, b_m) \mapsto b_m$ the product map $\phi_{\underline{v}}$ is also equivariant for the right action of B^m on $P_{\underline{v}}$ and the right multiplication of G by B . The product map therefore descends to a left B -equivariant morphism $f_{\underline{v}}: Z_{\underline{v}} \rightarrow X$.

Let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ the word obtained by dropping the rightmost reflection in \underline{v} . The projection map $\text{pr}_{\underline{v}}: P_{\underline{v}} \rightarrow P_{\underline{u}}$ sending (p_1, \dots, p_m) to (p_1, \dots, p_{m-1}) is equivariant with respect to the projection $B^m \rightarrow B^{m-1}$ sending (b_1, \dots, b_m) to (b_1, \dots, b_{m-1}) . Similarly the inclusion map $j_{\underline{v}}: P_{\underline{u}} \hookrightarrow P_{\underline{v}}$ sending (p_1, \dots, p_{m-1}) to $(p_1, \dots, p_{m-1}, 1_G)$ is equivariant with respect to the inclusion $B^{m-1} \hookrightarrow B^m$ sending (b_1, \dots, b_{m-1}) to $(b_1, \dots, b_{m-1}, b_{m-1})$. The maps $\text{pr}_{\underline{v}}$ and $j_{\underline{v}}$ respect the left B -action on $P_{\underline{v}}$ and $P_{\underline{u}}$, and therefore descend to B -equivariant maps $\pi_{\underline{v}}: Z_{\underline{v}} \rightarrow Z_{\underline{u}}$ and $\sigma_{\underline{v}}: Z_{\underline{u}} \rightarrow Z_{\underline{v}}$. Since $\text{pr}_{\underline{v}} \circ j_{\underline{v}} = \text{id}_{P_{\underline{u}}}$ and $\phi_{\underline{v}} \circ j_{\underline{v}} = \phi_{\underline{u}}$, taking quotients we obtain $\pi_{\underline{v}} \circ \sigma_{\underline{v}} = \text{id}_{Z_{\underline{u}}}$ and $f_{\underline{v}} \circ \sigma_{\underline{v}} = f_{\underline{u}}$. Finally, the fibres of $\pi_{\underline{v}}$ are isomorphic to $P_{\alpha_{i_m}}/B \cong \mathbf{P}^1$.

We record the following well-known facts about the construction above.

Proposition (2.11.2) —

- (a) The varieties Z_v produced by the recursive and direct constructions above are isomorphic over X .
- (b) If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is a reduced word with product v then the image of $f_{\underline{v}}: Z_{\underline{v}} \rightarrow X$ is X_v and $f_{\underline{v}}$ is a resolution of singularities of X_v .

Proof. Part (b) is proved in [De1] and [Ha]. To show (a) it is enough to show that the varieties produced by the direct construction satisfy the fibre product diagram (2.11.1). This is most easily checked after pulling back (2.11.1) via the maps $G \rightarrow X$ and $P_{\underline{u}} = P_{i_1} \times \cdots \times P_{i_{m-1}} \rightarrow Z_{\underline{u}}$; the details are omitted here. \square

Maximum points. Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a reduced word with product v . By Proposition 2.11.2(b) the image of $Z_{\underline{v}}$ under $f_{\underline{v}}$ is X_v and one can check that there is a unique point $p_{\underline{v}}$ of $Z_{\underline{v}}$ which maps to $v \in X_v$. More specifically, from the point of view of the direct construction the point $(s_{i_1}, \dots, s_{i_m})$ is a point of $P_{\underline{v}}$ and its image under the quotient map $P_{\underline{v}} \rightarrow Z_{\underline{v}}$ is $p_{\underline{v}}$. From the point of view of the recursive construction one starts with $p_\emptyset = e$, and recursively defines $p_{\underline{v}}$ to be unique torus fixed point in the \mathbf{P}^1 -fibre of $\pi_{\underline{v}}: Z_{\underline{v}} \rightarrow Z_{\underline{u}}$ over $p_{\underline{u}}$ which is not equal to $\sigma_{\underline{v}}(p_{\underline{u}})$, where $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$. Note that $p_{\underline{v}}$ is the unique torus fixed point of $Z_{\underline{v}}$ whose image in X_v is the largest in the Bruhat order among torus-fixed points of X_v . We call $p_{\underline{v}}$ the *maximum point* of $Z_{\underline{v}}$.

Since $p_{\underline{v}}$ is a torus fixed point, the torus acts on the tangent space $T_{p_{\underline{v}}} Z_{\underline{v}}$ and it will be important for us to know the formal character of $T_{p_{\underline{v}}} Z_{\underline{v}}$. It follows inductively from the recursive construction that

$$(2.11.3) \quad \text{Ch}(T_{p_v}Z_{\underline{v}}) = \langle \Phi_{v^{-1}} \rangle.$$

2.12. Semi-stability of torus fixed points. The following lemma is due to Kostant.

Lemma (2.12.1) — Let W be a projective variety with a G -action and L a G -equivariant ample line bundle on W . A torus fixed point $q \in W$ is semi-stable with respect to L if and only if the weight of L_q is zero. In this case the orbit of q is closed in the semi-stable locus.

Proof. If the action of the torus on the fibre L_q is non-trivial then it is easy to see (for instance using the Hilbert-Mumford criterion for semi-stability, [MFK, Theorem 2.1, p. 49]) that q is not a stable point.

Conversely, suppose that the weight of L_q is zero. Replacing L by a multiple we may assume that L is very ample and gives an embedding $W \hookrightarrow \mathbf{P}^r$ for some r . Let \mathbf{A}^{r+1} be the affine space corresponding to \mathbf{P}^r and $\mathbf{A}^{r+1} \setminus \{0\} \longrightarrow \mathbf{P}^r$ be the quotient map. Then G acts linearly on \mathbf{A}^{r+1} inducing an action on \mathbf{P}^r compatible with the action on W . Let \tilde{q} be any lift to \mathbf{A}^{r+1} of the image of q in \mathbf{P}^r . The condition that the torus act trivially on L_q is equivalent to the condition that \tilde{q} be fixed by T under the G -action on \mathbf{A}^{r+1} . Kostant ([K2, p. 354, Remark 11]) proves that for any finite dimensional module of a reductive group G and any point \tilde{q} fixed by T , the G -orbit of \tilde{q} is closed; this result was also later generalized by Luna [Lu2, Theorem (**)]. Since G is reductive and the orbit of \tilde{q} does not meet zero, there is a G -invariant homogeneous form of some degree m which is nonzero on \tilde{q} . This corresponds to a G -invariant section $s \in H^0(W, L^m)^G$ such that $s(q) \neq 0$. We thus see that if the weight of L_q is zero then q is a semi-stable point, and the orbit of q is closed in the semi-stable locus. \square

3. DIAGONAL BOTT-SAMELSON-DEMAZURE-HANSEN-KUMAR VARIETIES

In this section we give a generalization of the varieties from §2.11. The construction is a variation of a construction of Kumar in [Ku2]; see §3.10 for a comparison. These varieties are obtained by applying the idea of the Bott-Samelson resolution to the diagonal inclusion $X \hookrightarrow X^k$. They can also be thought of as a desingularization of the total space of the variety of intersections of translates of Schubert cycles. This alternate description is established in Theorem 3.7.4.

More specifically, for each sequence $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ of words we construct a smooth variety $Y_{\underline{v}}$ of dimension $N + \ell(\underline{v})$ with a G -action together with a proper map $f_{\underline{v}}: Y_{\underline{v}} \longrightarrow X^k$ which is G -equivariant for the diagonal action of G on X^k . If \underline{u} is the sequence obtained by dropping a single simple reflection from the right of one of the \underline{v}_j 's then $Y_{\underline{v}}$ is a \mathbf{P}^1 -fibration over $Y_{\underline{u}}$, and there is a section $Y_{\underline{u}} \hookrightarrow Y_{\underline{v}}$ compatible with the maps $f_{\underline{v}}$ and $f_{\underline{u}}$ to X^k . The fibration and section maps are G -equivariant; moreover they are compatible with the \mathbf{P}^1 -fibrations on factors of X^k . These relationships are summarized in diagram (3.1.2).

3.1. Recursive Construction. Let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. If all \underline{v}_j are empty, i.e., if $\underline{v} = (\emptyset, \dots, \emptyset)$, we set $Y_{\underline{v}} = X$ and let $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^k$ be the diagonal embedding.

Otherwise suppose that \underline{v}_j is nonempty. Let

$$(3.1.1) \quad \underline{u}_l := \begin{cases} \underline{v}_l & \text{if } l \neq j \\ (\underline{v}_j)_R & \text{if } l = j \end{cases}, \quad l = 1, \dots, k$$

and set $\underline{u} = (\underline{u}_1, \dots, \underline{u}_k)$. By induction on $\ell(\underline{v})$ we may assume that $Y_{\underline{u}}$ and the map $f_{\underline{u}}: Y_{\underline{u}} \rightarrow X^k$ have been constructed. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that $\underline{u}_j = s_{i_1} \cdots s_{i_{m-1}}$ then we define $Y_{\underline{v}}$, the map $f_{\underline{v}}$, the projection $\pi_{\underline{v}, \underline{u}}$, and the section $\sigma_{\underline{v}, \underline{u}}$ by the following fibre product square:

$$(3.1.2) \quad \begin{array}{ccccc} Y_{\underline{v}} & \xrightarrow{f_{\underline{v}}} & X^k & & \\ \downarrow \pi_{\underline{v}, \underline{u}} & \nearrow f_{\underline{u}} & \square & \downarrow (\text{id}_X)^{j-1} \times \pi_{i_m} \times (\text{id}_X)^{k-j} & \\ Y_{\underline{u}} & \xrightarrow{\quad} & X^{j-1} \times M_{i_m} \times X^{k-j} & & \end{array} .$$

Here $\pi_{i_m}: X \rightarrow M_{i_m} := G/P_{\alpha_{i_m}}$ is the natural projection, and $X^k \rightarrow X^{j-1} \times M_{i_m} \times X^{k-j}$ is the projection π_{i_m} on the j -th factor and the identity on all others. The bottom map $Y_{\underline{u}} \rightarrow X^{j-1} \times M_{i_m} \times X^{k-j}$ is the map $f_{\underline{u}}$ to X^k followed by the map $X^k \rightarrow X^{j-1} \times M_{i_m} \times X^{k-j}$ above.

Since $X^k \rightarrow X^{j-1} \times \pi_{i_m} \times X^{k-j}$ is a P^1 -fibration the same is true of $\pi_{\underline{v}, \underline{u}}$. We conclude by induction that the variety $Y_{\underline{v}}$ is smooth, proper, and irreducible of dimension $N + \ell(\underline{v})$. The maps $f_{\underline{u}}$ and $\text{id}_{Y_{\underline{u}}}$ from $Y_{\underline{u}}$ to X^k and $Y_{\underline{u}}$ respectively give rise to the section $\sigma_{\underline{v}, \underline{u}}$. By construction we have $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}, \underline{u}}$ and $\text{id}_{Y_{\underline{u}}} = \pi_{\underline{v}, \underline{u}} \circ \sigma_{\underline{v}, \underline{u}}$.

This construction is well-defined. Indeed, assume that we had dropped a simple reflection from the right of $\underline{v}_{j'}$, $j' \neq j$ to obtain a sequence of words \underline{u}' and used $Y_{\underline{u}'}$ instead of $Y_{\underline{u}}$ to construct $Y_{\underline{v}}$. We claim that the resulting variety $Y_{\underline{v}}$ is the same. This follows easily by induction on $\ell(\underline{v})$ and the fact that the diagram expressing the commutativity of the projections on the different factors is a fibre square:

$$\begin{array}{ccc} X^k & \xrightarrow{(\text{id}_X)^{j'-1} \times \pi_{i_m} \times (\text{id}_X)^{k-j'}} & X^{j'-1} \times M_{i'_m} \times X^{k-j'} \\ \downarrow (\text{id}_X)^{j-1} \times \pi_{i_m} \times (\text{id}_X)^{k-j} & \square & \downarrow \\ X^{j-1} \times M_{i_m} \times X^{k-j} & \longrightarrow & X^{j'-1} \times M_{i_{m'}} \times X^{j-j'-1} \times M_{i_m} \times X^{k-j} \end{array} .$$

Here, by symmetry, we have assumed that $j' < j$.

3.2. Direct Construction. Let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. The group B acts diagonally on $Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$ on the left. We define $Y_{\underline{v}}$ to be the quotient of $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k})$ by the left B -action

$$(3.2.1) \quad b \cdot (g, z_1, \dots, z_k) = (gb^{-1}, b \cdot z_1, \dots, b \cdot z_k).$$

Since $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k})$ is smooth and B acts without fixed points, the quotient $Y_{\underline{v}}$ is smooth.

The group G acts on $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k})$ by left multiplication on the first factor. Since this action commutes with the action of B , it descends to an action of G on $Y_{\underline{v}}$. The map from $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k})$ to X^k given by

$$(3.2.2) \quad (g, z_1, \dots, z_k) \mapsto (g \cdot f_{\underline{v}_1}(z_1), g \cdot f_{\underline{v}_2}(z_2), \dots, g \cdot f_{\underline{v}_k}(z_k))$$

is invariant under the B -action. If we let G act on X^k diagonally then (3.2.2) is also G -equivariant and hence descends to a G -equivariant morphism $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^k$.

As in the direct construction, we suppose that \underline{v}_j is nonempty, define \underline{u}_j by (3.1.1) and set $\underline{u} = (\underline{u}_1, \dots, \underline{u}_k)$. The B -equivariant morphisms $\pi_{\underline{v}}: Z_{\underline{v}_j} \rightarrow Z_{\underline{u}_j}$ and $\sigma_{\underline{v}_j}: Z_{\underline{u}_j} \rightarrow Z_{\underline{v}_j}$ from §2.11 give rise to B -equivariant morphisms between $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k})$ and $G \times (Z_{\underline{u}_1} \times \dots \times Z_{\underline{u}_k})$ and hence to a G -equivariant P^1 -fibration $\pi_{\underline{v}, \underline{u}}: Y_{\underline{v}} \rightarrow Y_{\underline{u}}$ and a G -equivariant section $\sigma_{\underline{v}, \underline{u}}: Y_{\underline{u}} \rightarrow Y_{\underline{v}}$. These maps fit together to give diagram (3.1.2).

3.3. Expanded Version of the Direct Construction. Combining the formulas for $P_{\underline{v}}$ from §2.11 with the direct construction above we obtain a more explicit expression for $Y_{\underline{v}}$. If $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ with $\underline{v}_j = s_{i_{1,j}} \cdots s_{i_{m_j,j}}$ for $j = 1, \dots, k$ then we define $Y_{\underline{v}}$ to be the quotient of

$$G \times P_{\underline{v}_1} \times \dots \times P_{\underline{v}_k} = G \times (P_{i_{1,1}} \times \dots \times P_{i_{m_1,1}}) \times \dots \times (P_{i_{1,k}} \times \dots \times P_{i_{m_k,k}})$$

by the right action of $B \times B^{m_1} \times \dots \times B^{m_k}$, where an element

$$(b_0 \mid b_{1,1}, \dots, b_{m_1,1} \mid b_{1,2}, \dots, b_{m_2,2} \mid \dots \mid b_{1,k}, \dots, b_{m_k,k})$$

acts from the right on

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \dots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \dots, p_{i_{m_2,2}} \mid \dots \mid p_{i_{1,k}}, \dots, p_{i_{m_k,k}})$$

to give

$$(gb_0 \mid b_0^{-1}p_{i_{1,1}}b_{1,1}, b_{1,1}^{-1}p_{i_{2,1}}b_{2,1}, \dots, b_{m_1-1,1}^{-1}p_{i_{m_1,1}}b_{m_1,1} \mid \dots \mid b_0^{-1}p_{i_{1,k}}b_{1,k}, \dots, b_{m_k-1,k}^{-1}p_{i_{m_k,k}}b_{m_k,k}).$$

(In the expressions above the vertical lines “|” are used to indicate logical groupings, but otherwise have no significance.) The group G acts on $G \times P_{\underline{v}_1} \times \dots \times P_{\underline{v}_k}$ by left multiplication on the G factor, this action descends to a left action on $Y_{\underline{v}}$.

The map $f_{\underline{v}}$ is induced by the map sending an element

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \dots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \dots, p_{i_{m_2,2}} \mid \dots \mid p_{i_{1,k}}, \dots, p_{i_{m_k,k}})$$

of $G \times P_{\underline{v}_1} \times \dots \times P_{\underline{v}_k}$ to

$$(3.3.1) \quad (gp_{i_{1,1}}p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid gp_{i_{1,2}}p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \dots \mid gp_{i_{1,k}} \cdots p_{i_{m_k,k}})$$

in X^k . From the explicit formulas this is clearly a G -equivariant map.

Finally, if \underline{v} is a sequence of words, and \underline{u} is a sequence obtained by dropping the right-most reflection of a single word in \underline{v} (as in §3.2) then the G -equivariant P^1 -fibration $\pi_{\underline{v}, \underline{u}}: Y_{\underline{v}} \rightarrow Y_{\underline{u}}$ and the G -equivariant section $\sigma_{\underline{v}, \underline{u}}: Y_{\underline{u}} \rightarrow Y_{\underline{v}}$ are constructed using the obvious formulas analogous to those in §2.11. It again follows easily from these formulas that $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}, \underline{u}}$.

Remark. Note that the variety $Y_{\underline{v}}$ depends on the sequence of words $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ and not just on the corresponding sequence (v_1, \dots, v_k) of Weyl group elements. If we choose a different reduced factorization of each v_i the resulting variety is birational to $Y_{\underline{v}}$ over X^k . The proof is omitted because we do not need this fact.

3.4. The map f_{\circ} . As before, let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. Besides the map $f_{\underline{v}}$ to X^k , each $Y_{\underline{v}}$ comes with a G -equivariant map f_{\circ} to X expressing $Y_{\underline{v}}$ as a $Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$ -bundle over X .

From the point of view of the construction in §3.1 f_{\circ} is the composite map

$$Y_{\underline{v}} \xrightarrow{\sigma_{\underline{v}, \underline{u}}} Y_{\underline{u}} \longrightarrow \dots \longrightarrow Y_{\emptyset} = X$$

obtained by dropping the elements in the entries of \underline{v} one at a time. The fibre over e in X is then the result of applying the recursive construction in §2.11 separately for each \underline{v}_i , $i = 1, \dots, k$, and so the fibre is $Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$.

From the point of view of the construction in §3.2 one starts with the projection $G \times (Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}) \rightarrow G$ onto the first factor. This is B -equivariant for the right action of B on G and hence descends to a morphism $f_{\circ}: Y_{\underline{v}} \rightarrow X$ expressing $Y_{\underline{v}}$ as a $Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$ -bundle over X .

Let $\underline{u} = (\emptyset, \underline{v}_1, \dots, \underline{v}_k)$. Since the action of B on the point $Z_{\emptyset} = e$ is trivial, we have an isomorphism

$$G \times Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k} \simeq G \times Z_{\emptyset} \times Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$$

of B -varieties and hence a G -isomorphism $\phi: Y_{\underline{v}} \rightarrow Y_{\underline{u}}$. From the explicit description in (3.2.2) we see that the composite map $f_{\underline{u}} \circ \phi: Y_{\underline{v}} \rightarrow X^{k+1}$ followed by projection onto the first factor is f_{\circ} , and that $f_{\underline{u}} \circ \phi$ followed by projection onto the last k factors is $f_{\underline{v}}$.

Thus the map $f_{\circ} \times f_{\underline{v}}: Y_{\underline{v}} \rightarrow X \times X^k$ is equal to the map $f_{(\emptyset, \underline{v}_1, \dots, \underline{v}_k)}: Y_{(\emptyset, \underline{v}_1, \dots, \underline{v}_k)} \rightarrow X^{k+1}$ under the isomorphism ϕ . This will be used in the proof of Theorem 3.7.4.

3.5. Maximum point. Let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of words. We define the *maximum point* $p_{\underline{v}}$ of $Y_{\underline{v}}$ to be the product maximum point (§2.11) $p_{\underline{v}_1} \times \dots \times p_{\underline{v}_k}$ in the fibre $Z_{\underline{v}_1} \times \dots \times Z_{\underline{v}_k}$ of f_{\circ} over e in X . Alternatively, if $\underline{v}_j = (s_{i_{1,j}}, \dots, s_{i_{m_j,j}})$ for $j = 1, \dots, k$ then (in the notation of §3.3) the point

$$(e \mid s_{i_{1,1}}, s_{i_{2,1}}, \dots, s_{i_{m_1,1}} \mid \dots \mid s_{i_{1,k}}, \dots, s_{i_{m_k,k}})$$

is a point of

$$G \times (P_{i_{1,1}} \times \dots \times P_{i_{m_1,1}}) \times \dots \times (P_{i_{1,k}} \times \dots \times P_{i_{m_k,k}})$$

and its image in $Y_{\underline{v}}$ under the quotient map by $B \times B^{m_1} \times \dots \times B^{m_k}$ is the maximum point $p_{\underline{v}}$. If each \underline{v}_j is a factorization of some $v_j \in \mathcal{W}$, then the image $f_{\underline{v}}(p_{\underline{v}})$ of the maximum point in X^k is the point $q_{\underline{v}} := (v_1, \dots, v_k)$.

3.6. Tangent space formulas. We will need to know the formal character (see §2.1) of the tangent space of $Y_{\underline{v}}$ at the maximum point $p_{\underline{v}}$. If each \underline{v}_j is a reduced word with product v_j , then the formal character of the tangent space to $Z_{\underline{v}_j}$ at $p_{\underline{v}_j}$ is $\langle \Phi_{v_j^{-1}} \rangle$ and the formal character of the tangent space of X at e is $\langle \Delta^- \rangle$.

Since the fibration f_{\circ} is smooth, the formal character of $T_{p_{\underline{v}}} Y_{\underline{v}}$ is the sum of these formal characters, i.e.,

$$\text{Ch}(T_{p_{\underline{v}}} Y_{\underline{v}}) = \langle \Delta^- \rangle + \sum_{i=1}^k \langle \Phi_{v_i^{-1}} \rangle.$$

If $v_j = w_j^{-1}w_0$ for $j = 1, \dots, k$, then by (2.2.2) this is the same as

$$(3.6.1) \quad \text{Ch}(T_{p_{\underline{v}}} Y_{\underline{v}}) = \Delta^- + \sum_{i=1}^k \langle \Phi_{w_i}^c \rangle.$$

3.7. Fibres and images of $f_{\underline{v}}$.

Lemma (3.7.1) — Let $\underline{v} = (\emptyset, \underline{v}_2, \dots, \underline{v}_k)$ be a sequence of words, with each \underline{v}_i a reduced factorization of v_i , and let $X_{\underline{v}}$ be the (reduced) image of $f_{\underline{v}}$ in X^k . Then:

- (a) Projection onto the first factor of X^k endows $X_{\underline{v}}$ with the structure of a fibre bundle over X with fibre isomorphic to $X_{v_2} \times \dots \times X_{v_k}$.
- (b) The variety $X_{\underline{v}}$ is normal with rational singularities of dimension $N + \ell(\underline{v})$, and the induced map $Y_{\underline{v}} \longrightarrow X_{\underline{v}}$ is birational with connected fibres.

Proof. Projection on the first factor of X^k gives a G -equivariant morphism $X_{\underline{v}} \xrightarrow{\eta} X$. Since G acts transitively on X this morphism is surjective and all fibres are isomorphic, i.e., this expresses $X_{\underline{v}}$ as a fibre bundle over X . To study the fibres we look at the fibre $\eta^{-1}(e)$ over the B -fixed point e of X .

Consider the diagram

$$(3.7.2) \quad \begin{array}{ccc} G \times e \times Z_{v_2} \times \cdots \times Z_{v_k} & \xrightarrow{\psi_{\underline{v}}} & Y_{\underline{v}} \\ id_G \times id_{x_o} \times f_{v_1} \times \cdots \times f_{v_k} \downarrow & & \downarrow f_{\underline{v}} \\ G \times e \times X_{v_2} \times \cdots \times X_{v_k} & \xrightarrow{\phi} & X^k \end{array}$$

where ϕ is given by $\phi(g, e, x_2, \dots, x_k) = (g \cdot e, g \cdot x_2, \dots, g \cdot x_k) \in X^k$. Since $\psi_{\underline{v}}$ and the leftmost vertical map are surjective, the image of $f_{\underline{v}}$ is the same as the image of ϕ . Since B is the stabilizer of e , the fibre $\eta^{-1}(e)$ is the image of $B \times e \times X_{v_2} \times \cdots \times X_{v_k}$ under ϕ . But each Schubert variety X_w is stable under the action of B and therefore the image above is just $e \times X_{v_2} \times \cdots \times X_{v_k}$, proving (a).

From the fibration η it is clear that

$$\dim(X_{\underline{v}}) = \dim(X) + \sum_{i=2}^k \dim(X_{v_i}) = N + \sum_{i=2}^k \ell(v_i) = N + \ell(\underline{v})$$

because each v_i is reduced and hence $\ell(v_i) = \ell(v_i)$ for $i \geq 2$.

The product of normal varieties is again normal, and the product of varieties with rational singularities also has rational singularities. Since each X_w is normal with rational singularities (§2.6), the fibres also have this property, and therefore so does $X_{\underline{v}}$ (since the properties of being normal or having rational singularities are local, and $X_{\underline{v}}$ is locally the product of the fibre and a smooth variety).

Since each map $f_{v_i}: Z_{v_i} \rightarrow X_{v_i}$ is a resolution of singularities of a normal variety, each f_{v_i} is birational with connected fibres. It follows that the map $Y_{\underline{v}} \rightarrow X_{\underline{v}}$, which is the quotient of the leftmost vertical map in (3.7.2) by the action of B is also birational with connected fibres. This proves (b). \square

Definition (3.7.3) — If X_w is any Schubert subvariety of X and q is any point of X , we define the subvariety qX_w of X to be the result of translating X_w by any element in the B -coset corresponding to q . Since X_w is B -stable the result is independent of the choice of representative for q .

The following theorem gives more precise information about the image and fibres of $f_{\underline{v}}$.

Theorem (3.7.4) — Let $\underline{v} = (v_1, \dots, v_k)$ be a sequence of reduced words with corresponding Weyl group elements (v_1, \dots, v_k) . Then there exists a factorization $f_{\underline{v}}: Y_{\underline{v}} \xrightarrow{\tau} Q_{\underline{v}} \xrightarrow{h} X^k$ such that

- (a) $Q_{\underline{v}}$ is normal with rational singularities;
- (b) the map $\tau: Y_{\underline{v}} \rightarrow Q_{\underline{v}}$ is proper and birational with connected fibres;
- (c) for each point (q_1, \dots, q_k) of X^k there is a natural inclusion of the scheme-theoretic fibre $h^{-1}(q_1, \dots, q_k)$ into the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$;

- (d) the inclusion of schemes in (c) induces an isomorphism at the level of reduced schemes, or in other words, the set theoretic fibre $h^{-1}(q_1, \dots, q_k)$ is equal to the set theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$.

Proof. Let $f_\circ \times f_{\underline{v}}: Y_{\underline{v}} \longrightarrow X \times X^k$ be the product of $f_{\underline{v}}$ and the map $f_\circ: Y_{\underline{v}} \longrightarrow X$ from §3.4 expressing $Y_{\underline{v}}$ as a $Z_{v_1} \times \dots \times Z_{v_k}$ -bundle over X . We define $Q_{\underline{v}}$ to be the image of $f_\circ \times f_{\underline{v}}$ with the reduced scheme structure, τ to be the map from $Y_{\underline{v}}$ onto $Q_{\underline{v}}$, and h to be the map from $Q_{\underline{v}}$ to X^k induced by the projection $X \times X^k \longrightarrow X^k$. By construction $f_{\underline{v}} = h \circ \tau$.

Letting $\psi_{\underline{v}}$ be the map (from §3.2) defining $Y_{\underline{v}}$ as a quotient of $B \times Z_{v_1} \times \dots \times Z_{v_k}$ and $\phi: G \times X_{v_1} \times \dots \times X_{v_k} \longrightarrow Q_{\underline{v}} \subseteq X \times X^k$ as the map sending (g, x_1, \dots, x_k) to $(gB/B, g \cdot x_1, \dots, g \cdot x_k)$ in $X \times X^k$, we obtain a refinement of diagram (3.7.2):

$$\begin{array}{ccccc}
 G \times Z_{v_1} \times \dots \times Z_{v_k} & \xrightarrow{\psi_{\underline{v}}} & Y_{\underline{v}} & & \\
 \downarrow & \square & \downarrow \tau & \searrow f_\circ \times f_{\underline{v}} & \\
 G \times X_{v_1} \times \dots \times X_{v_k} & \xrightarrow{\phi} & Q_{\underline{v}} & \hookrightarrow & X \times X^k \\
 \downarrow h & & \downarrow & & \downarrow \\
 X^k & \xlongequal{\quad} & X^k & &
 \end{array}$$

Since $Y_{\underline{v}} \simeq Y_{(\emptyset, v_1, \dots, v_k)}$ (see §3.4) and under this isomorphism the map $f_\circ \times f_{\underline{v}}$ is the map $f_{(\emptyset, v_1, \dots, v_k)}$, it follows from Lemma 3.7.1(b) that $Q_{\underline{v}}$ is normal with rational singularities and that $\tau: Y_{\underline{v}} \longrightarrow Q_{\underline{v}}$ is birational with connected fibres, proving (a) and (b).

The composite map

$$G \times P_{v_1} \times \dots \times P_{v_k} \longrightarrow G \times Z_{v_1} \times \dots \times Z_{v_k} \xrightarrow{\psi_{\underline{v}}} Y_{\underline{v}} \xrightarrow{\tau} Q_{\underline{v}}$$

is given (in the notation of §3.3) by sending

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \dots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \dots, p_{i_{m_2,2}} \mid \dots \mid p_{i_{1,k}}, \dots, p_{i_{m_k,k}})$$

to

$$(g \mid gp_{i_{1,1}}p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid gp_{i_{1,2}}p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \dots \mid gp_{i_{1,k}} \cdots p_{i_{m_k,k}})$$

in $X \times X^k$. A point q of X is therefore in the fibre $h^{-1}(q_1, \dots, q_k) \subseteq X \times q_1 \times \dots \times q_k = X$ if for any B -coset representatives g, g_1, \dots, g_k of q, q_1, \dots, q_k , there exist elements $\{p_{i,j}\}$ in the respective parabolic subgroups such that we can solve the equations

$$\begin{aligned}
gp_{i_{1,1}}p_{i_{2,1}} \cdots p_{i_{m_1,1}} &= g_1 \\
&\vdots &&\vdots &\vdots \\
gp_{i_{1,k}}p_{i_{2,k}} \cdots p_{i_{m_k,k}} &= g_k.
\end{aligned}$$

Moving the $p_{i,j}$'s to the right hand side, the system above becomes

$$\begin{aligned}
g &= g_1 p_{i_{m_1,1}}^{-1} \cdots p_{i_{2,1}}^{-1} p_{i_{1,1}}^{-1} \\
&\vdots &&\vdots &\vdots \\
g &= g_k p_{i_{m_k,k}}^{-1} \cdots p_{i_{2,k}}^{-1} p_{i_{1,k}}^{-1}
\end{aligned}$$

which is equivalent to q belonging in the intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$, proving (d).

Let \underline{v} be a reduced word with product v . By part (d) the set

$$Q'_{\underline{v}} := \{(q, p) \in X \times X \mid p \in qX_v\}$$

is the image of $f_{(\emptyset, \underline{v})}: Y_{(\emptyset, \underline{v})} \rightarrow X \times X$ and is therefore a closed subvariety of $X \times X$. Alternatively $Q'_{\underline{v}}$ is the Zariski closure of the set $\{(g, g \cdot v) \mid g \in G\} \subseteq X \times X$.

For $i = 1, \dots, k$, let $p_i: X \times X^k \rightarrow X \times X$ be the map which is the product of id_X with projection $X^k \rightarrow X$ onto the i -th factor. The intersection

$$Q'_{\underline{v}} := \bigcap_{i=1}^k p_i^{-1}(Q'_{\underline{v}_i})$$

is a closed subscheme of $X \times X^k$ which, by (d), agrees set theoretically with $Q_{\underline{v}}$. Since $Q_{\underline{v}}$ is reduced, we have the inclusion of schemes $Q_{\underline{v}} \subseteq Q'_{\underline{v}}$. If h' is the map $h': Q'_{\underline{v}} \rightarrow X^k$ induced by projection, then the scheme-theoretic fibres of h are naturally a subscheme of the scheme-theoretic fibres of h' (and both are naturally subschemes of X). The scheme-theoretic fibre of h' is the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$, proving (c). \square

The image $X_{\underline{v}}$ is therefore the set of translations (q_1, \dots, q_k) in X^k for which the intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ of translated Schubert varieties is non-empty, and the set-theoretic fibres of h are the intersections themselves. Moreover, $Q_{\underline{v}}$ is the incidence correspondence of intersections of translates of Schubert varieties (the first coordinate in $X \times X^k$ is the intersection, the remaining k coordinates are the parameters (q_1, \dots, q_k) controlling the translates). Theorem 3.7.4 shows that $Y_{\underline{v}}$ is a resolution of singularities of $Q_{\underline{v}}$.

Corollary (3.7.5) — Let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ be a sequence of reduced words with corresponding Weyl group elements (v_1, \dots, v_k) such that $\sum_{i=1}^k \ell(\underline{v}_i) = (k-1)N$. Then the degree of the map $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^k$ is given by the intersection number $\bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^k [X_{v_i^{-1}}]$.

Remark. The dimension of $Y_{\underline{v}}$ in this case is $N + \sum \ell(\underline{v}_i) = kN = \dim(X^k)$ so it is reasonable to ask for the degree of the map.

Proof. Since we are working in characteristic zero, the degree of $f_{\underline{v}}$ is given by the number of points in a generic fibre. By Theorem 3.7.4 the map $p: Y_{\underline{v}} \rightarrow Q_{\underline{v}}$ is birational, and so the generic fibre of $f_{\underline{v}}$ is the same as the generic fibre of $h: Q_{\underline{v}} \rightarrow X^k$. By the Kleiman transversality theorem, if q_1, \dots, q_k are generic, the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced and finite, and the number of points is equal to the intersection number $\bigcap_{i=1}^k [X_{v_i^{-1}}] = \bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}]$ in $H^*(X, \mathbf{Z})$. By Theorem 3.7.4(c–d) if the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced it is equal to the scheme-theoretic fibre $h^{-1}(q_1, \dots, q_k)$, proving the corollary. \square

3.8. Key Lemma. We now prove an important lemma which will allow us to derive several results necessary for the proofs of Theorems I and II. The lemma itself will also be used in the proof of Theorem I.

Lemma (3.8.1) — Let \underline{v} be a sequence of reduced words, L be a G -equivariant line bundle on $Y_{\underline{v}}$ and $s \in H^0(Y_{\underline{v}}, L)^G$ be a nonzero G -invariant section. Then

- (a) the weight of L at the T -fixed maximum point ($\S 3.5$) $p = p_{\underline{v}} \in Y_{\underline{v}}$ belongs to $\text{span}_{\mathbf{Z}_{\geq 0}} \Delta^+$;
- (b) the weight of L at p is zero if and only if s does not vanish at p ;
- (c) without supposing that L has a G -invariant section, if L is an equivariant bundle on $Y_{\underline{v}}$ and the weight of L at p is zero, then $\dim H^0(Y_{\underline{v}}, L)^G \leq 1$.

Remark. Part (c) will be used repeatedly to control the size of the G -invariant sections.

Proof. Let $f_{\circ}: Y_{\underline{v}} \rightarrow X$ be the map from $\S 3.4$ expressing $Y_{\underline{v}}$ as a $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ -bundle over X . The section s cannot vanish on any fibre of f_{\circ} since (by G -invariance and transitivity of G -action on X) s would vanish on all of $Y_{\underline{v}}$. We can thus restrict s to get a nonzero section on the fibre $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ of f_{\circ} over $e \in X$; this fibre contains the maximum point p .

The formal character of the tangent space at the maximum point p_i of $Z_{\underline{v}_i}$ is $\langle \Phi_{v_i^{-1}} \rangle$; i.e., all the weights of this space are positive roots. Since the maximum point $p = p_1 \times \cdots \times p_k \in Z := Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ is the product of the maximum points of the factors, each of the weights on the tangent space of p in Z is also a positive root.

Let \mathfrak{m}_p be the maximal ideal of p in $\mathcal{O}_{Z,p}$. For every $r \geq 0$ we get a T -equivariant restriction map

$$H^0(Z, L|_Z) \longrightarrow L \otimes_{\mathcal{O}_Z} (\mathcal{O}_{Z,p}/\mathfrak{m}_p^{r+1}) = L \otimes (\mathcal{O}_Z/\mathfrak{m}_p \oplus \mathfrak{m}_p/\mathfrak{m}_p^2 \oplus \cdots \oplus \mathfrak{m}_p^r/\mathfrak{m}_p^{r+1})$$

which is an injection for r sufficiently large. In particular, for sufficiently large r , the section s restricts to a nonzero element of $L \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,p}/\mathfrak{m}_p^{r+1}$. Since s is an invariant section,

this means that the zero weight is a weight of $L \otimes_{\mathcal{O}_Z} (\mathcal{O}_{Z,p}/\mathfrak{m}_p^{r+1})$, and so must appear in one of the factors $L \otimes_{\mathcal{O}_Z} (\mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}) = L \otimes_{\mathcal{O}_Z} \text{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ for $i = 0, \dots, r$.

Since $\mathfrak{m}_p/\mathfrak{m}_p^2$ is dual to the tangent space at p , all weights of $\mathfrak{m}_p/\mathfrak{m}_p^2$ are negative roots, and therefore the weights of $\text{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ belong to $\text{span}_{\mathbf{Z} \leq 0} \Delta^+$. Tensoring with L multiplies the formal character of $\text{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ by the weight of L at p . Thus the zero weight is a weight of $L \otimes_{\mathcal{O}_Z} \text{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ only if the weight of L at p belongs to $\text{span}_{\mathbf{Z} \geq 0} \Delta^+$. This proves (a).

The value of s at p is the restriction of s to the factor $L \otimes_{\mathcal{O}_Z} (\mathcal{O}_{Z,p}/\mathfrak{m}_p) = L_p$. If s does not vanish at p the weight of L_p is therefore zero. Conversely, if the weight of L_p is zero then the weights of $L \otimes_{\mathcal{O}_Z} \text{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ are non-zero for $i \geq 1$. Hence the only possibility for the invariant section s under the restriction map is to have nonzero restriction to $L \otimes_{\mathcal{O}_Z} (\mathcal{O}_{Z,p}/\mathfrak{m}_p) = L_p$, proving (b).

Suppose that the weight of L at p is zero. If there were two linearly independent sections $s_1, s_2 \in H^0(Y_{\underline{v}}, L)^G$ then some nonzero linear combination would vanish at p contradicting (b). Hence if the weight is zero we must have $\dim H^0(Y_{\underline{v}}, L) \leq 1$, giving (c). \square

3.9. Applications of Lemma 3.8.1.

Theorem (3.9.1) — Suppose that w_1, \dots, w_k , and w are elements of the Weyl group such that $\ell(w) = \sum_{i=1}^k \ell(w_i)$ and $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$ in $H^*(X, \mathbf{Z})$. Then:

- (a) For any dominant weights μ_1, \dots, μ_k , and μ such that the irreducible module V_μ is a component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$, the weight $\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu$ belongs to $\text{span}_{\mathbf{Z} \geq 0} \Delta^+$.
- (b) If $\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu = 0$ then $\text{mult}(V_\mu, V_{\mu_1} \otimes \dots \otimes V_{\mu_k}) = 1$.
- (c) $\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 = \sum_{i=1}^k (w_i^{-1} \rho - \rho) - (w^{-1} \rho - \rho)$ belongs to $\text{span}_{\mathbf{Z} \geq 0} \Delta^+$.
- (d) If $\sum_{i=1}^k w_i^{-1} \cdot 0 = w^{-1} \cdot 0$ then $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$.

Note that the action of the Weyl group in parts (a) and (b) is the homogeneous action, while the action in parts (c) and (d) is the affine action.

Proof. Let $v_i = w_i^{-1} w_0$ for $i = 1, \dots, k$, $v_{k+1} = w^{-1}$, let \underline{v}_i be a reduced word with product v_i , for $i = 1, \dots, k+1$, and set $\underline{v} = (\underline{v}_1, \dots, \underline{v}_{k+1})$. Then $\sum \ell(\underline{v}_i) = (k+1-1)N$ and so, by Corollary 3.7.5, the degree of $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^{k+1}$ is given by the intersection number

$$\bigcap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w].$$

By hypothesis this intersection number is nonzero and therefore $f_{\underline{v}}$ is surjective.

Given dominant weights μ_1, \dots, μ_k , and μ let $\lambda_i = -w_0\mu_i$ for $i = 1, \dots, k$ and $\lambda_{k+1} = \mu$. Set L to be the line bundle $L_{\lambda_1} \boxtimes \dots \boxtimes L_{\lambda_{k+1}}$ on X^{k+1} , so that $H^0(X^{k+1}, L) = V_{\mu_1} \otimes \dots \otimes V_{\mu_k} \otimes V_{\mu}^*$ and $\dim H^0(X^{k+1}, L)^G$ is the multiplicity of V_{μ} in the tensor product. $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$.

Since $f_{\underline{Y}}$ is surjective, pullback induces an inclusion

$$H^0(Y_{\underline{Y}}, f_{\underline{Y}}^*L) \xleftarrow{f_{\underline{Y}}^*} H^0(X^{k+1}, L)$$

and, in particular, $\dim H^0(Y_{\underline{Y}}, f_{\underline{Y}}^*L)^G \geq \dim H^0(X^{k+1}, L)^G$. Applying Lemma 3.8.1(a), we know that if $f_{\underline{Y}}^*L$ has a nonzero G -invariant section then the weight of $f_{\underline{Y}}^*L$ at the maximum point $p_{\underline{Y}}$ belongs to $\text{span}_{\mathbf{Z}_{\geq 0}} \Delta^+$. This weight is

$$(3.9.2) \quad \sum_{i=1}^{k+1} v_i(-\lambda_i) = \sum_{i=1}^k (w_i^{-1} w_0)(w_0 \mu_i) + w^{-1}(-\mu) = \sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu,$$

proving (a).

If the weight in (3.9.2) is zero then $\dim H^0(X^{k+1}, L)^G \leq \dim H^0(Y_{\underline{Y}}, f_{\underline{Y}}^*L)^G \leq 1$ by Lemma 3.8.1(c), and so if V_{μ} is a component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$ then it is of multiplicity at most one. The fact that V_{μ} actually is a component of the tensor product is a consequence of the solution of the PRV conjecture – see §2.3 for a discussion. This proves (b).

The map $f_{\underline{Y}}: Y_{\underline{Y}} \rightarrow X^{k+1}$ induces a natural map $f_{\underline{Y}}^*K_{X^{k+1}} \rightarrow K_{Y_{\underline{Y}}}$ which is given by a global section s of $H^0(Y_{\underline{Y}}, (f_{\underline{Y}}^*K_{X^{k+1}})^* \otimes K_{Y_{\underline{Y}}})$. Since $Y_{\underline{Y}}$ and X^{k+1} have the same dimension and since $f_{\underline{Y}}$ is surjective this section is nonzero. Because the pullback morphism is natural, the section s is G -invariant. By Lemma 3.8.1(a) the weight of the line bundle $K_{Y_{\underline{Y}}/X^{k+1}} := (f_{\underline{Y}}^*K_{X^{k+1}})^* \otimes K_{Y_{\underline{Y}}}$ at the maximum point $p_{\underline{Y}}$ belongs to $\text{span}_{\mathbf{Z}_{\geq 0}} \Delta^+$.

By (3.6.1), (2.2.2), and (2.2.3) the formal characters of the tangent spaces at $p_{\underline{Y}}$ in $Y_{\underline{Y}}$ and $q_{\underline{Y}} := f_{\underline{Y}}(p_{\underline{Y}})$ in X^{k+1} are, respectively:

$$(3.9.3) \quad \text{Ch}(T_{p_{\underline{Y}}} Y_{\underline{Y}}) = \langle \Phi_w \rangle + \langle \Delta^- \rangle + \sum_{i=1}^k \langle \Phi_{w_i}^c \rangle$$

and

$$(3.9.4) \quad \text{Ch}(T_{q_{\underline{Y}}} X^{k+1}) = (\langle \Phi_w \rangle + \langle -\Phi_w^c \rangle) + \sum_{i=1}^k (\langle \Phi_{w_i}^c \rangle + \langle -\Phi_{w_i} \rangle).$$

A short calculation using formula (2.2.4) shows that the weight of $K_{Y_{\underline{Y}}/X^{k+1}}$ at $p_{\underline{Y}}$ is $\sum_{i=1}^k (w_k^{-1} \rho - \rho) - (w^{-1} \rho - \rho)$, proving (c).

If the weight $\sum_{i=1}^k (w_k^{-1} \rho - \rho) - (w^{-1} \rho - \rho)$ is zero then, by Lemma 3.8.1(b), the section s is nonzero at $p_{\underline{Y}}$. This means that $f_{\underline{Y}}$ is unramified at $p_{\underline{Y}}$ and therefore the tangent space map $T_{Y_{\underline{Y}}, p} \xrightarrow{df_{\underline{Y}}} T_{X^{k+1}, q}$ is an isomorphism. Hence both spaces must have the same formal

characters. Comparing the negative roots and their multiplicities in (3.9.3) and (3.9.4) gives $\Delta^- = (\sqcup_{i=1}^k -\Phi_{w_i}) \sqcup -\Phi_w^c$ which is equivalent to $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$, proving (d). \square

3.10. Relation with existing results. Part (a) of Theorem (3.9.1) is due to Berenstein and Sjamaar. A theorem of this type was first proved by Klyacho [Kly] for GL_n . This was later extended to all semisimple groups by Berenstein and Sjamaar in [BeSj] and by Kapovich, Leeb, and Milson in [KLM]. Parts (c) and (d) are due to Belkale and Kumar: part (c) is [BK, Theorem 29] and (d) is [BK, Theorem 15], both in the case when the parabolic group P is the Borel group B .

Part (b) is new and crucial for controlling the multiplicities of cohomological components. The remaining statements have been included because Lemma 3.8.1 allows us to give a new, short, and unified proof of these results. In particular, we obtain a new proof of the necessity of the inequalities determining the Littlewood-Richardson cone. Namely, these inequalities are obtained by requiring that the weights in Theorem 3.9.1(a) (for all w_1, \dots, w_k, w satisfying the conditions of the theorem) belong to $\text{span}_{\mathbf{Z} \geq 0} \Delta^+$. (The proof that these inequalities are sufficient requires a separate GIT argument.)

Relation with a construction of Kumar. Given a sequence \underline{u} of simple reflections, Kumar [Ku2, §1.1] defined a variety $\tilde{Z}_{\underline{u}}$ along with a map $\theta_{\underline{u}}$ from $\tilde{Z}_{\underline{u}}$ to X^2 . For any pair of words $\underline{v} = (\underline{v}_1, \underline{v}_2)$ let $\underline{u} = \underline{v}_1^{-1} \underline{v}_2$ be the word obtained by reversing \underline{v}_1 and concatenating it onto the left of \underline{v}_2 . By comparing the construction of $Y_{\underline{v}}$ and $\tilde{Z}_{\underline{u}}$ it is not hard to find an isomorphism $\tilde{Z}_{\underline{u}} = Y_{\underline{v}}$ over X^2 (i.e., such that $\theta_{\underline{u}} = f_{\underline{v}}$ under the isomorphism). Therefore when $k = 2$ the varieties produced by our construction are the same the ones constructed in [Ku2, §1.1].

4. PROOF OF THEOREM III

4.1. We will prove Theorem III in its symmetric form. After applying the symmetrization procedure from §2.8 (and replacing $k + 1$ by k) we obtain:

Theorem (4.1.1) — (Symmetric form of Theorem III) Let w_1, \dots, w_k be elements of \mathcal{W} group such that $\sum_i \ell(w_i) = N$; $\lambda_1, \dots, \lambda_k$ be weights such that $w_i \cdot \lambda_i$ are dominant weights for $i = 1, \dots, k$; and $\sum_{i=1}^k \lambda_i = -2\rho$.

(a) If $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ then the cup product map

$$(4.1.2) \quad H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^N(X, K_X)$$

is surjective.

(b) If $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$ then (4.1.2) is zero.

The proof of Theorem 4.1.1 is given in §4.3. We will use the following common notation. For any sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ of weights let $L_{\underline{\lambda}}$ be the line bundle

$$L_{\underline{\lambda}} := L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_k} = pr_1^* L_{\lambda_1} \otimes \cdots \otimes pr_k^* L_{\lambda_k}$$

on X^k , where $pr_i: X^k \rightarrow X$ denotes projection onto the i -th factor.

4.2. Inductive Lemma. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a sequence of weights and $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ a sequence of words. Let \underline{u} be a sequence of words as in (3.1.1), i.e., \underline{u} is a sequence of words obtained by dropping a simple reflection from the right of a single member of \underline{v} . The following lemma lets us propagate information about the pullback map

$$(4.2.1) \quad H^{N+\ell(\underline{v})}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}}) \xleftarrow{f_{\underline{v}}^*} H^{N+\ell(\underline{v})}(X^k, L_{\underline{\lambda}})$$

on the top degree cohomology of $Y_{\underline{v}}$ to information about an analogous pullback map to the top degree cohomology of $Y_{\underline{u}}$. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that we are dropping s_{i_m} from \underline{v}_j to get \underline{u}_j , we denote by $\underline{\mu}$ the sequence $\underline{\mu} := (\lambda_1, \dots, \lambda_{j-1}, s_{i_m} \cdot \lambda_j, \lambda_{j+1}, \dots, \lambda_k)$. Finally, we assume that the degree of $L_{\underline{\lambda}}$ is negative on the fibres of the P^1 -fibration $\pi_{\underline{v}, \underline{u}}: Y_{\underline{v}} \rightarrow Y_{\underline{u}}$.

Lemma (4.2.2) — Under the conditions above, the pullback map

$$H^{N+\ell(\underline{u})}(Y_{\underline{u}}, f_{\underline{u}}^* L_{\underline{\mu}}) \xleftarrow{f_{\underline{u}}^*} H^{N+\ell(\underline{u})}(X^k, L_{\underline{\mu}})$$

is (a) surjective, (b) zero, or (c) surjective on the space of G -invariants if the pullback map (4.2.1) has the corresponding property (a), (b), or (c).

Here surjective on the space of G -invariants means (in the case of $Y_{\underline{v}}$) that

$$H^{N+\ell(\underline{v})}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}})^G \xleftarrow{f_{\underline{v}}^*} H^{N+\ell(\underline{v})}(X^k, L_{\underline{\lambda}})^G$$

is surjective.

Proof. To reduce notation set $M_{\underline{u}} = X^{j-1} \times M_{i_m} \times X^{k-j}$ and let $\pi: X^k \rightarrow M_{\underline{u}}$ be the map $\pi = (\text{id}_X)^{j-1} \times \pi_{i_m} \times (\text{id}_X)^{k-j}$. The fibre product diagram (3.1.2) relating $Y_{\underline{v}}$, $Y_{\underline{u}}$, X^k , and $M_{\underline{u}}$ is

$$(4.2.3) \quad \begin{array}{ccc} Y_{\underline{v}} & \xrightarrow{f_{\underline{v}}} & X^k \\ \sigma_{\underline{v}} \downarrow \pi_{\underline{v}} \square & & \downarrow \pi \\ Y_{\underline{u}} & \xrightarrow{h} & M_{\underline{u}} \end{array}$$

where $h = \pi \circ f_{\underline{u}}$ and where we use $\pi_{\underline{v}}$ and $\sigma_{\underline{v}}$ in place of $\pi_{\underline{v}, \underline{u}}$ and $\sigma_{\underline{v}, \underline{u}}$ to reduce notation.

Note that $L_{\underline{\mu}}$ is the Demazure reflection of $L_{\underline{\lambda}}$ with respect to π . By §2.9 this means that we have natural isomorphisms

$$(4.2.4) \quad \pi_{\underline{v}*}(f_{\underline{v}}^* L_{\underline{\mu}}) \cong R^1 \pi_{\underline{v}*}(f_{\underline{v}}^* L_{\underline{\lambda}}) \quad \text{and} \quad \pi_* L_{\underline{\mu}} \cong R^1 \pi_* L_{\underline{\lambda}}$$

valid on $Y_{\underline{u}}$ and $M_{\underline{u}}$ respectively. Diagram (4.2.3), the Leray spectral sequences for $L_{\underline{\lambda}}$ and $L_{\underline{\mu}}$ relative to π and $\pi_{\underline{v}}$, and the isomorphisms (4.2.4) then give the commutative diagram of cohomology groups:

$$(4.2.5) \quad \begin{array}{ccc} H^{N+\ell(\underline{v})}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}}) & \xleftarrow[f_{\underline{v}}^*]{(4.2.1)} & H^{N+\ell(\underline{v})}(X^k, L_{\underline{\lambda}}) \\ \downarrow \text{Leray} & & \downarrow \text{Leray} \\ H^{N+\ell(\underline{v})-1}(Y_{\underline{u}}, R^1 \pi_{\underline{v}*} f_{\underline{v}}^* L_{\underline{\lambda}}) & \xleftarrow{h^*} & H^{N+\ell(\underline{v})-1}(M_{\underline{u}}, R^1 \pi_{\underline{v}*} L_{\underline{\lambda}}) \\ \downarrow \text{(4.2.4)} & & \downarrow \text{(4.2.4)} \\ H^{N+\ell(\underline{v})-1}(Y_{\underline{u}}, \pi_{\underline{v}*} f_{\underline{v}}^* L_{\underline{\mu}}) & \xleftarrow{h^*} & H^{N+\ell(\underline{v})-1}(M_{\underline{u}}, \pi_{\underline{v}*} L_{\underline{\mu}}) \\ \downarrow \text{Leray} & & \downarrow \text{Leray} \\ H^{N+\ell(\underline{v})-1}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\mu}}) & \xleftarrow[f_{\underline{v}}^*]{} & H^{N+\ell(\underline{v})-1}(X^k, L_{\underline{\mu}}) \end{array}$$

We conclude that the bottom pullback map $H^{N+\ell(\underline{v})-1}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\mu}}) \xleftarrow[f_{\underline{v}}^*]{} H^{N+\ell(\underline{v})-1}(X^k, f_{\underline{v}}^* L_{\underline{\mu}})$ is surjective, zero, or surjective on the space of G -invariants if (4.2.1) is.

On $Y_{\underline{v}}$ we have the exact sequence of bundles:

$$(4.2.6) \quad 0 \longrightarrow f_{\underline{v}}^* L_{\underline{\mu}}(-Y_{\underline{u}}) \longrightarrow f_{\underline{v}}^* L_{\underline{\mu}} \longrightarrow f_{\underline{v}}^* L_{\underline{\mu}}|_{Y_{\underline{u}}} \longrightarrow 0,$$

where we consider $Y_{\underline{u}}$ to be a divisor in $Y_{\underline{v}}$ via the section $\sigma_{\underline{v}}$. The degree of $f_{\underline{v}}^* L_{\underline{\mu}}(-Y_{\underline{u}})$ is at least -1 on the fibres of $\pi_{\underline{v}}$ so the corresponding Leray spectral sequence gives

$$H^{N+\ell(\underline{v})}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\mu}}(-Y_{\underline{u}})) = H^{N+\ell(\underline{v})}\left(Y_{\underline{u}}, \pi_{\underline{v}*}(f_{\underline{v}}^* L_{\underline{\mu}}(-Y_{\underline{u}}))\right) = 0$$

where the second cohomology group above equals zero by reason of dimension:

$$N + \ell(\underline{v}) = N + \ell(\underline{u}) + 1 = \dim(Y_{\underline{u}}) + 1.$$

The end of the long exact cohomology sequence associated to (4.2.6) is therefore

$$(4.2.7) \quad H^{N+\ell(\underline{v})-1}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{v}}^*} H^{N+\ell(\underline{v})-1}(Y_{\underline{u}}, f_{\underline{v}}^* L_{\underline{\mu}}|_{Y_{\underline{u}}}) \longrightarrow 0.$$

Since $\ell(\underline{u}) = \ell(\underline{v}) - 1$, $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}}$, and all maps are G -equivariant, we conclude that the pullback map $f_{\underline{u}}^*$, being the composite map

$$H^{N+\ell(\underline{u})}(X^k, L_{\underline{\mu}}) \xrightarrow{f_{\underline{v}}^*} H^{N+\ell(\underline{u})}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{v}}^*} H^{N+\ell(\underline{u})}(Y_{\underline{u}}, f_{\underline{v}}^* L_{\underline{\mu}}|_{Y_{\underline{u}}}) = H^{N+\ell(\underline{u})}(Y_{\underline{u}}, f_{\underline{u}}^* L_{\underline{\mu}}),$$

is (a) surjective, (b) zero, or (c) surjective on the space of G -invariants, if the pullback map $f_{\underline{v}}^*$ in (4.2.1) has the corresponding property (a), (b), or (c). \square

Remark. In part (c) of Lemma 4.2.2 we can replace the statement about G -invariants with a statement about any isotypic component; the proof above goes through without change, we will only need the case of G -invariants as part of the proof of Theorem I in §5 below.

4.3. Proof of Theorem 4.1.1 and variation. For the rest of this section, we fix the following notation. Let w_1, \dots, w_k and $\lambda_1, \dots, \lambda_k$ be as in Theorem 4.1.1. For each $i = 1, \dots, k$ set $v_i := w_i^{-1}w_0$ and $\lambda'_i := v_i^{-1} \cdot \lambda_i$. Let \underline{v}_i be a reduced factorization of v_i and let $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$. Finally, set $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ and $\underline{\lambda}' = (\lambda'_1, \dots, \lambda'_k)$.

Proof of Theorem 4.1.1. Since

$$\dim(Y_{\underline{v}}) = N + \sum_{i=1}^k \ell(v_i) = N + \sum_{i=1}^k (N - \ell(w_i)) = kN = \dim(X^k),$$

Corollary 3.7.5 implies that the degree of $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^k$ is given by the intersection number $\bigcap_{i=1}^k [\Omega_{w_i}]$. Therefore the pullback map $H^{kN}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}'}) \xleftarrow{f_{\underline{v}}^*} H^{kN}(X^k, L_{\underline{\lambda}'})$ is a surjection if $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ and is zero if $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$. Indeed, if $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ then $f_{\underline{v}}$ is a birational map between the smooth varieties $Y_{\underline{v}}$ and X^k in characteristic zero, and so the pullback map $H^j(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}'}) \xleftarrow{f_{\underline{v}}^*} H^j(X^k, L_{\underline{\lambda}'})$ is an isomorphism in all degrees, and in particular is a surjection in degree $j = kN$. On the other hand, if $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$ then the image $X_{\underline{v}}$ of $f_{\underline{v}}$ is subvariety of X^k of dimension strictly less than kN and therefore the pullback map $f_{\underline{v}}^*$ in top cohomology, which factors through $H^{kN}(X_{\underline{v}}, L_{\underline{\lambda}}|_{X_{\underline{v}}}) = 0$, is the zero map.

Consider a sequence

$$\underline{v} =: \underline{v}^0, \underline{v}^1, \dots, \underline{v}^{(k-1)N} := \underline{\emptyset} = (\emptyset, \dots, \emptyset)$$

of sequences of words which reduces \underline{v} to the empty sequence, and where at each step \underline{v}^{j+1} is obtained by dropping a simple reflection from the right of a single member of \underline{v}^j . Set $\underline{\lambda}^j = (\underline{v}^j)^{-1} \cdot \underline{\lambda}$ where (by slight abuse of notation) \underline{v}^j is considered as an element of \mathcal{W}^k and the action is componentwise. Note that $\underline{\lambda}^0 = \underline{\lambda}'$ and $\underline{\lambda}^{(k-1)N} = \underline{\lambda}$. The construction of \underline{v}^j and $\underline{\lambda}^j$ implies that the degree of $L_{\underline{\lambda}^j}$ is negative on the fibres of the P^1 -fibration $\pi_{\underline{v}^j, \underline{v}^{j+1}}: Y_{\underline{v}^j} \rightarrow Y_{\underline{v}^{j+1}}$.

Applying Lemma 4.2.2 to the pairs $(\underline{v}^j, \underline{v}^{j+1})$ for $j = 0, \dots, (k-1)N - 1$ we conclude that

$$(4.3.1) \quad H^N(Y_{\underline{\emptyset}}, f_{\underline{\emptyset}}^* L_{\underline{\lambda}}) \xleftarrow{f_{\underline{\emptyset}}^*} H^N(X^k, L_{\underline{\lambda}})$$

is surjective if $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ and zero if $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$. By construction $f_{\underline{\emptyset}}: Y_{\underline{\emptyset}} = X \rightarrow X^k$ is the diagonal embedding of X into X^k and the pullback map (4.3.1) is the cup product map. This proves Theorem 4.1.1 and completes the proof of Theorem III. \square

We record a statement that will be used in the proof of Theorem I below.

Proposition (4.3.2) — If the pullback map

$$H^{kN}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}'}) \xleftarrow{f_{\underline{v}}^*} H^{kN}(X^k, L_{\underline{\lambda}'})$$

is surjective on the space of G -invariants then the cup product map (4.1.2) is surjective.

Proof. We repeat the inductive reduction in the proof of Theorem 4.1.1 above with part (c) of Lemma 4.2.2 in place of parts (a) and (b). As a result we conclude that the cup product

map (4.1.2) is surjective on the space of G -invariants. Since $H^N(X, K_X)$ is the trivial G -module we conclude that (4.1.2) is surjective. \square

5. PROOF OF THEOREM I AND COROLLARIES

In this section we use Theorem III and Proposition 4.3.2 to prove Theorem I. The proof that (1.2.1) is necessary for the surjectivity of the cup product map appears in §5.1 and the proof that (1.2.1) is sufficient appears in §5.3.

5.1. Proof that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ is a necessary condition for surjectivity. We assume the notation of §1.2, and set $\mu_i = w_i \cdot \lambda_i$ for $i = 1 \dots k$, and $\mu = w \cdot \lambda$. By assumption the weights μ_1, \dots, μ_k , and μ are dominant. By the Borel-Weil-Bott theorem each $H^{\ell(w_i)}(X, L_{\lambda_i}) = V_{\mu_i}^*$ and $H^d(X, L_{\lambda}) = V_{\mu}^*$.

Since $w_i^{-1}\mu_i = w_i^{-1} \cdot \mu_i - w_i^{-1} \cdot 0$ and $w^{-1}\mu_i = w^{-1} \cdot \mu_i - w^{-1} \cdot 0$, we have

$$\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = \left(\sum_{i=1}^k w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu \right) - \left(\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 \right).$$

Furthermore $\sum_{i=1}^k w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu = \sum \lambda_i - \lambda = 0$ and so the equation above becomes

$$(5.1.1) \quad \sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = - \left(\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 \right).$$

If the cup product map $H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^d(X, L_{\lambda})$ is surjective, then (after dualizing) V_{μ} must be a component of the tensor product $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ and by Theorem III(b), the intersection $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$ in $H^*(X, \mathbf{Z})$; we may therefore apply Theorem 3.9.1.

By Theorem 3.9.1(a) the left hand side of (5.1.1) belongs to $\text{span}_{\mathbf{Z} \geq 0} \Delta^+$ and by part (c) of the same theorem the right hand side belongs to $\text{span}_{\mathbf{Z} \leq 0} \Delta^+$. We conclude that both sides are zero and so by Theorem 3.9.1(d) that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$. \square

Remark. In the first half of the argument above the hypothesis that the cup product map is surjective was used, along with Theorem III, to conclude that V_{μ} is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ and $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$. If, on the other hand, we assume the latter two conditions then the second half of the argument still applies to give $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$. We will use this observation in Corollary 5.4.7 below.

5.2. Setup for the proof of sufficiency. For convenience, we collect some of the consequences of condition (1.2.1) in its symmetric form which have effectively appeared in previous arguments, and which we will use in the proof of sufficiency.

Proposition (5.2.1) — Suppose that w_1, \dots, w_k are elements of the Weyl group such that $\Delta^+ = \sqcup_{i=1}^k \Phi_{w_i}$.

Combinatorial Consequences:

- (a) $\sum_{i=1}^k w_i^{-1} \cdot 0 = -2\rho$.
- (b) Suppose that $\lambda_1, \dots, \lambda_k$ are weights such that $\sum \lambda_i = -2\rho$, and set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \dots, k$. Then $\sum_{i=1}^k w_i^{-1} \mu_i = 0$.

Geometric Consequences: For each $i = 1, \dots, k$, let $v_i = w_i^{-1} w_0$ and let \underline{v}_i be a word which is a reduced factorization of v_i . We set $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$ and construct as usual the variety $Y_{\underline{v}}$ and the map $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^k$.

Then

- (c) $\deg(f_{\underline{v}}) \neq 0$.
- (d) The weight of the relative canonical bundle $K_{Y_{\underline{v}}/K_{X^k}}$ at $p_{\underline{v}}$ is zero.

Proof. Part (a) is immediate from the condition $\Delta^+ = \sqcup_{i=1}^k \Phi_{w_i}$ and formula (2.2.4). Part (b) reverses the argument used to arrive at (5.1.1) in §5.1:

$$\sum_{i=1}^k \mu^{-1} \mu_i = \left(\sum_{i=1}^k w_i^{-1} \cdot \mu_i \right) - \left(\sum_{i=1}^k w_i^{-1} \cdot 0 \right) = \sum_{i=1}^k \lambda_i - (-2\rho) = 0.$$

Part (c) is Corollary 3.7.5 combined with Lemma 2.6.1. Part (d) is the symmetric version of the computation in the proof of Theorem 3.9.1(d): the weight of the relative canonical bundle $K_{Y_{\underline{v}}/X^k}$ at $p_{\underline{v}}$ is $\sum_{i=1}^k w_i^{-1} \cdot 0 + 2\rho$ which is zero by part (a). \square

5.3. Proof that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ is a sufficient condition for surjectivity. Consider the symmetric version of the problem as in §2.8. It suffices to show the surjectivity of a cup product map

$$(5.3.1) \quad H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^N(X, K_X)$$

where w_1, \dots, w_k are elements of the Weyl group such that $\sum \ell(w_i) = N$; $\lambda_1, \dots, \lambda_k$ are weights such that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \dots, k$ and $\sum \lambda_i = -2\rho$. After this reduction condition (1.2.1) becomes $\Delta^+ = \sqcup_{i=1}^k \Phi_{w_i}$. We recall the notation from §4.3: $v_i := w_i^{-1} w_0$, $\lambda'_i := v_i^{-1} \cdot \lambda_i$, \underline{v}_i is a reduced factorization of v_i , $\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$, and $\underline{\lambda}' = (\lambda'_1, \dots, \lambda'_k)$.

By Proposition 4.3.2 to show the surjectivity of (5.3.1) it is enough to show that the pull-back map

$$(5.3.2) \quad H^{kN}(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\lambda}'})^G \xleftarrow{f_{\underline{v}}^*} H^{kN}(X^k, L_{\underline{\lambda}'})^G$$

on the space of G -invariants is surjective. We will show that both spaces of G -invariants are one-dimensional, and that the induced map is an isomorphism. Note that by Proposition 5.2.1(c) $\deg(f_{\underline{v}}) \neq 0$ and so $f_{\underline{v}}$ is surjective.

The pullback map on top cohomology is Serre dual to the trace map:

$$H^0(Y_{\underline{v}}, (f_{\underline{v}}^* L_{\underline{\lambda}'})^* \otimes K_{Y_{\underline{v}}}) = H^0 \left(Y_{\underline{v}}, f_{\underline{v}}^*(L_{\underline{\lambda}'}^* \otimes K_{X^k}) \otimes K_{Y_{\underline{v}}/X^k} \right) \xrightarrow{\text{Tr}_{f_{\underline{v}}}} H^0(X^k, L_{\underline{\lambda}'}^* \otimes K_{X^k}).$$

Let $s \in H^0(Y_{\underline{v}}, K_{Y_{\underline{v}}/X^k})^G$ be the nonzero G -invariant section giving the map $f_{\underline{v}}^* K_{X^k} \longrightarrow K_{Y_{\underline{v}}}$ induced by $f_{\underline{v}}$, and let E be the divisor of $s = 0$. The composition

$$\begin{aligned} H^0(X^k, L_{\underline{\lambda}'}^* \otimes K_{X^k}) &\xrightarrow{f_{\underline{v}}^*} H^0 \left(Y_{\underline{v}}, f_{\underline{v}}^*(L_{\underline{\lambda}'}^* \otimes K_{X^k}) \right) \\ &\xrightarrow{\cdot E} H^0 \left(Y_{\underline{v}}, f_{\underline{v}}^*(L_{\underline{\lambda}'}^* \otimes K_{X^k}) \otimes K_{Y_{\underline{v}}/X^k} \right) \xrightarrow{\text{Tr}_{f_{\underline{v}}}} H^0(X^k, L_{\underline{\lambda}'}^* \otimes K_{X^k}) \end{aligned}$$

of pullback, multiplication by E , and the trace map is multiplication by $\deg(f_{\underline{v}})$, which is nonzero. This gives us the inequality

$$(5.3.3) \quad \dim H^0(X^k, L_{\underline{\lambda}'}^* \otimes K_{X^k})^G \leq \dim H^0 \left(Y_{\underline{v}}, f_{\underline{v}}^*(L_{\underline{\lambda}'}^* \otimes K_{X^k}) \otimes K_{Y_{\underline{v}}/X^k} \right)^G$$

and shows that in order to prove that the trace map induces an isomorphism on G -invariants it is sufficient to prove that we have equality of dimensions in (5.3.3).

Set $\mu_i = w_i \cdot \lambda_i = w_0 \cdot \lambda'_i$ for $i = 1, \dots, k$. By the Borel-Weil-Bott Theorem we have $H^{kN}(X^k, L_{\underline{\lambda}'}) = V_{\mu_1}^* \otimes \cdots \otimes V_{\mu_k}^*$ and so (by Serre duality) $H^0(X^k, L_{\underline{\lambda}'}^* \otimes K_{X^k}) = V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. Now set $\nu_i = -w_0 \mu_i$ for $i = 1, \dots, k$ so that $V_{\nu_i} = V_{\mu_i}^*$ and let $\underline{\nu} = (\nu_1, \dots, \nu_k)$. By the calculation

$$S(\lambda'_i) = -\lambda'_i - 2\rho = -w_0 \cdot \mu_i - 2\rho = -(w_0 \mu_i - 2\rho) - 2\rho = -w_0 \mu_i$$

in each coordinate factor (as in §2.5) we conclude that $L_{\underline{\lambda}'}^* \otimes K_{X^k} = L_{\underline{\nu}}$.

The weight of $L_{\underline{\nu}}$ at $q := f_{\underline{v}}(p_{\underline{v}}) = (v_1, \dots, v_k)$ is

$$-\sum_{i=1}^k v_i \nu_i = \sum_{i=1}^k (w_i^{-1} w_0)(w_0 \mu_i) = \sum_{i=1}^k w_i^{-1} \mu_i \stackrel{5.2.1(b)}{=} 0.$$

Since by Proposition 5.2.1(d) the weight of $K_{Y_{\underline{v}}/X^k}$ at $p_{\underline{v}}$ is zero, the weight of $f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k}$ at $p_{\underline{v}}$ in $Y_{\underline{v}}$ is also zero and hence $\dim H^0(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k})^G \leq 1$ by Lemma 3.8.1(c). On the other hand, Lemma 2.3.1 implies that $(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G \neq 0$ so we conclude that $\dim H^0(X^k, L_{\underline{\nu}})^G \geq 1$. This gives us

$$1 \leq \dim H^0(X^k, L_{\underline{\nu}})^G \leq \dim H^0(Y_{\underline{v}}, f_{\underline{v}}^* L_{\underline{\nu}} \otimes K_{Y_{\underline{v}}/X^k})^G \leq 1.$$

Therefore the inequality in (5.3.3) is an equality, and the cup product map in (5.3.1) is surjective. \square

5.4. Corollaries of Theorem I and its proof.

Corollary (5.4.1) — The cup product map $H^0(X, L_{\lambda_1}) \otimes H^d(X, L_{\lambda_2}) \longrightarrow H^d(X, L_{\lambda_1} \otimes L_{\lambda_2})$ is surjective whenever both sides are nonzero.

Proof. If w_2 is the element of the Weyl group so that $w_2 \cdot \lambda_2 \geq 0$ then the conditions that λ_1 is dominant and that $L_{\lambda_1 + \lambda_2}$ has cohomology in the same degree d as L_{λ_2} imply that $w_2 \cdot (\lambda_1 + \lambda_2) \geq 0$, and so the corollary follows from Theorem I and the obvious statement that $\Phi_{w_2} = \Phi_{w_2} \sqcup \Phi_e$. \square

Corollary (5.4.2) — (Compatibility with Leray Spectral Sequence). Suppose that λ_1, λ_2 , and $\lambda = \lambda_1 + \lambda_2$ are regular weights and that the cup product map

$$H^{d_1}(X, L_{\lambda_1}) \otimes H^{d_2}(X, L_{\lambda_2}) \xrightarrow{\cup} H^d(X, L_\lambda)$$

is nonzero. Let P be any parabolic subgroup of G containing B , and $\pi: X \longrightarrow M := G/P$ be the corresponding projection. Then the cup product map on X factors as a composition

$$\begin{array}{ccc} H^{d_1}(X, L_{\lambda_1}) \otimes H^{d_2}(X, L_{\lambda_2}) & \xrightarrow{\cup_X} & H^d(X, L_\lambda) \\ \parallel & & \parallel \\ H^{d_1-i}(M, R_{\pi*}^i L_{\lambda_1}) \otimes H^{d_2-j}(M, R_{\pi*}^j L_{\lambda_2}) & \xrightarrow{\cup_M} & H^{d-i-j}(M, R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2}) \xrightarrow{\cup_\pi} H^{d-i-j}(M, R_{\pi*}^{i+j} L_\lambda) \end{array}$$

of the cup product on M followed by the map induced on cohomology by the relative cup product map $R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2} \longrightarrow R_{\pi*}^{i+j} L_\lambda$ on the fibres of π . A similar statement holds for the cup product of an arbitrary number of factors.

Proof. The factorization statement amounts to a numerical condition on the cohomology degrees of the line bundles on the fibres of π ensuring that the cup product map is computed by the map on E_2 -terms of the Leray spectral sequence. This numerical condition is immediately implied by (1.2.1). We explain this in more detail below.

Set $L = L_{\lambda_1} \boxtimes L_{\lambda_2}$ on $X \times X$ and consider the following factorization of the diagonal map $\delta_X: X \hookrightarrow X \times X$:

$$(5.4.3) \quad \begin{array}{ccccc} X & \xrightarrow{s} & X \times_M X & \xrightarrow{t} & X \times X \\ \downarrow \pi & & \downarrow \psi & & \downarrow \pi \times \pi \\ M & \xlongequal{\quad} & M & \xrightarrow{\delta_M} & M \times M \end{array} \quad \square$$

The cup product map then factors as

$$(5.4.4) \quad H^d(X, L_\lambda) \xleftarrow{s^*} H^d(X \times_M X, t^* L) \xleftarrow{t^*} H^{d_1}(X, L_{\lambda_1}) \otimes H^{d_2}(X, L_{\lambda_2}) = H^d(X \times X, L)$$

and we claim that (5.4.4) induces the factorization claimed above.

By the Borel-Weil-Bott theorem applied to the fibres of π , for each of the line bundles $L_{\lambda_1}, L_{\lambda_2}$, and L_λ there is precisely one degree for which the higher direct image sheaf is nonzero. Suppose i is the degree such that $R_{\pi*}^i L_{\lambda_1} \neq 0$, j is the degree such that $R_{\pi*}^j L_{\lambda_2} \neq 0$, and k is the degree such that $R_{\pi*}^k L_\lambda \neq 0$. The Leray spectral sequence for

the cohomology of these bundles degenerates at the E_2 term and we have the isomorphisms $H^{d_1}(X, L_{\lambda_1}) = H^{d_1-i}(M, R_{\pi*}^i L_{\lambda_1})$, $H^{d_2}(X, L_{\lambda_2}) = H^{d_2-j}(M, R_{\pi*}^j L_{\lambda_2})$, and $H^d(X, L_{\lambda_1}) = H^{d-k}(M, R_{\pi*}^k L_{\lambda_1})$.

Since $R_{\pi \times \pi*}^{i+j} L = R_{\pi*}^i L_{\lambda_1} \boxtimes R_{\pi*}^j L_{\lambda_2}$ is a vector bundle on $M \times M$, the theorem on cohomology and base change gives us $R_{\psi*}^{i+j} t^* L = \delta_M^*(R_{\pi*}^i L_{\lambda_1} \boxtimes R_{\pi*}^j L_{\lambda_2}) = R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2}$ on M and therefore we have $H^d(X \times_M X, t^* L) = H^{d-i-j}(M, R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2})$. The Leray spectral sequences for L and $t^* L$ with respect to ψ and $\pi \times \pi$ also degenerate at the E_2 -terms and have nonzero terms in the same degree. The discussion in §2.10 implies that the map on E_2 -terms computes the pullback map t^* . Therefore t^* in (5.4.4) is equal to the map

$$H^{d-i-j}(M, R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2}) \xleftarrow{\delta_M^*} H^{d_1-i}(M, R_{\pi*}^i L_{\lambda_1}) \otimes H^{d_2-j}(M, R_{\pi*}^j L_{\lambda_2})$$

which shows that t^* is the first part of the factorization claimed.

We now study s^* . The map s includes X as the relative diagonal of $X \times_M X$ over M . It follows that s^* induces the relative cup product map on the higher direct image sheaves of $t^* L$ and L_{λ} . Therefore the map associated to s^* on the E_2 -terms of the Leray spectral sequences for $t^* L$ and L_{λ} is given by the relative cup product map

$$H^{d-i-j}(M, R^{i+j} L_{\lambda}) = H^{d-i-j}(M, R^{i+j}(L_{\lambda_1} \otimes L_{\lambda_2})) \xleftarrow{\cup_{\pi}} H^{d-i-j}(M, R_{\pi*}^i L_{\lambda_1} \otimes R_{\pi*}^j L_{\lambda_2}).$$

All that is needed to demonstrate the factorization claimed is to demonstrate the condition $k = i + j$ which ensures the map on the associated graded pieces in the E_2 -terms agrees with the global map on the cohomology groups (c.f. §2.10).

Suppose that w_1 , w_2 , and w are the elements of the Weyl group such that $w_1 \cdot \lambda_1$, $w_2 \cdot \lambda_2$, and $w \cdot \lambda$ are dominant. Then $k = \#(\Phi_w \cap -\Delta_P)$, $i = \#(\Phi_{w_1} \cap -\Delta_P)$, and $j = \#(\Phi_{w_2} \cap -\Delta_P)$ where the symbol $\#$ indicates the cardinality of a set. The condition $k = i + j$ guaranteeing the factorization thus amounts to the condition

$$(5.4.5) \quad \#(\Phi_w \cap -\Delta_P) = \#(\Phi_{w_1} \cap -\Delta_P) + \#(\Phi_{w_2} \cap -\Delta_P).$$

Since the original cup product map was assumed surjective we must have $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ by Theorem I; this immediately implies that (5.4.5) holds. \square

Corollary (5.4.6) — Suppose that w_1, \dots, w_k are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$, and that μ_1, \dots, μ_k are dominant weights satisfying the condition $\sum_{i=1}^k w_i^{-1} \mu_i = 0$. Then $\dim(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G = 1$.

Proof. Set $\lambda_i = w_i^{-1} \cdot \mu_i$ for $i = 1, \dots, k$. Then $\sum \lambda_i = -2\rho$ and we have a cup product problem as in (5.3.1). As part of the proof of Theorem I in §5.3 it was established that $\dim(V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})^G = 1$. Alternatively, the corollary is simply Theorem 3.9.1(b) applied in symmetric form, with Lemma 2.6.1 used to ensure that the hypotheses of the theorem are satisfied. \square

Corollary (5.4.7) — Suppose that we have a cup product map

$$H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^d(X, L_\lambda)$$

and as above Weyl group elements w_1, \dots, w_k , and w such that $\mu_i := w_i \cdot \lambda_i$, $i = 1, \dots, k$, and $\mu := w \cdot \mu$ are dominant weights. Then if $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$ the cup product map is surjective if and only if V_μ is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$.

Proof. If V_μ is not a component of the tensor product the map is clearly not surjective. Conversely, if V_μ is a component, the assumption on the intersection number and the argument in §5.1 for the necessity of condition (1.2.1) show that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$, and therefore we conclude that the map is surjective by the sufficiency of condition (1.2.1). \square

The following example illustrates Corollary 5.4.7 and provides an example which shows that condition (1.2.1) is not necessary in order to have a cup product problem for which both sides are nonzero.

Example (5.4.8) — Let $G = \mathrm{SL}_6$ and $w_1 = w_2 = s_2 s_4 s_3$. For any integers $a_i, b_i \geq 0$ ($i = 1, 2$) set $\mu_i = (0, a_i, 0, b_i, 0)$ and $\lambda_i = w_i^{-1} \cdot \mu_i = (a_i + 1, b_i + 1, -4 - a_i - b_i, a_i + 1, b_i + 1)$. (The weights are written in terms of the fundamental weights of SL_6 .) Finally, let $w = s_1 s_3 s_5 s_2 s_4 s_3$ and set $\mu = w \cdot (\lambda_1 + \lambda_2) = (0, a_1 + a_2 + 1, 0, b_1 + b_2 + 1, 0) \in \Lambda^+$. We therefore get a cup product problem:

$$H^3(X, L_{\lambda_1}) \otimes H^3(X, L_{\lambda_2}) \xrightarrow{\cup} H^6(X, L_{\lambda_1 + \lambda_2}).$$

This cup product cannot be surjective by Theorem I since $\Phi_{w_1} = \Phi_{w_2}$; alternatively the map cannot be surjective since V_μ is clearly not a component of $V_{\mu_1} \otimes V_{\mu_2}$. The intersection number $([\Omega_{w_1}] \cap [\Omega_{w_2}]) \cdot X_w$ is two.

Corollary (5.4.9) — If $\Delta^+ = \sqcup_{i=1}^k \Phi_{w_i}$ then for any subset $I \subseteq \{1, \dots, k\}$ there is an element w of the Weyl group such that $\Phi_w = \sqcup_{i \in I} \Phi_{w_i}$.

Proof. Let $\lambda_i = w_i^{-1} \cdot 0$ so that we get a cup product problem as in (5.3.1). (Here each $H^{\ell(w_i)}(X, L_{\lambda_i})$ is the trivial G -module). By Theorem I and the assumption on w_1, \dots, w_k this cup product is surjective. It can be factored by first taking the cup product of any subset $I \subseteq \{1, \dots, k\}$ of the factors and the resulting cup product problem must also be nonzero since the larger problem is. Hence by Theorem I there is a $w \in \mathcal{W}$ with $w \cdot (\sum_{i \in I} \lambda_i) \in \Lambda^+$ and such that $\Phi_w = \sqcup_{i \in I} \Phi_{w_i}$. \square

5.5. Comments.

1. Corollary 5.4.9 can also be proved independently of any of the constructions in this paper by using a similar argument in nilpotent cohomology. We are grateful to Olivier Mathieu for pointing this out to us.
2. Using the result of Corollary 5.4.9 and induction, to prove Theorem I it is sufficient to prove it in the case $k = 2$ of the cup product of two cohomology groups into a third. We have chosen to develop the description of the varieties Y_v for arbitrary k partly since this

is the natural generality of the construction, partly because it makes no difference in our proofs, but also because some of the applications (e.g., the multiplicity bounds) do not follow by induction. Note that by the methods of this paper, even to prove the case $k = 2$ of the cup product it would be necessary to consider the case of the cup product of three factors into $H^N(X, K_X)$, and hence we would need the construction of $Y_{\underline{v}}$ for three factors.

3. As Example 5.4.8 shows, the natural numerical condition $\ell(w_1) + \ell(w_2) = \ell(w)$ does not imply condition (1.2.1) even if there is a nontrivial cup product problem corresponding to w_1, w_2 , and w . On the other hand, Condition (5.4.5) imposes further necessary numerical conditions for (1.2.1). Namely,

$$(5.5.1) \quad \ell(w_1^P) + \ell(w_2^P) = \ell(w^P) \text{ for every parabolic subgroup } P \supseteq B \text{ of } G,$$

where w_1^P, w_2^P, w^P denote the minimal length representatives in $w_1\mathcal{W}_P, w_2\mathcal{W}_P, w\mathcal{W}_P$. In the case when $G = \mathrm{SL}_{n+1}$ one can show that condition (5.5.1) is sufficient for (1.2.1). The simple inductive argument relies on the fact that if $G = \mathrm{SL}_{n+1}$ it is possible to assign a parabolic $P_\alpha \supset B$ to every root $\alpha \in \Delta^+$ in such a way that $-\alpha$ is a root of P_α but not a root of any proper parabolic subgroup of P_α containing B . We do not know if (5.5.1) is sufficient to imply (1.2.1) for general G .

4. Corollary 5.4.2 establishes the following Factorization Property: any nonzero cup product map on X factors as a cup product on G/P and fibres of $\pi: X \rightarrow G/P$ for all $P \supset B$. We know of no apriori reason why this should hold. The Factorization Property is equivalent to (5.4.5) holding for all $P \supset B$ which is equivalent to (5.5.1). Hence, in the case $G = \mathrm{SL}_{n+1}$ the Factorization Property is equivalent to (1.2.1).

6. COHOMOLOGICAL COMPONENTS AND PROOF OF THEOREM II

6.1. Conditions on components of tensor products.

We begin by introducing two relevant conditions. We also recall the notion of generalized PRV component from §2.3 for convenience.

Definitions (6.1.1) — Suppose that μ_1, \dots, μ_k , and μ are dominant weights.

- (a) We say that V_μ is a *generalized PRV component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist w_1, \dots, w_k , and w in \mathcal{W} such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$.
- (b) We say that V_μ is a component of *stable multiplicity one* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if we have $\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k} \otimes V_{m\mu}^*)^G = 1$ for all $m \gg 0$.
- (c) We say that V_μ is a *cohomological component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist w_1, \dots, w_k , and w in \mathcal{W} such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ and such that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$.

Under the hypothesis that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ the condition $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ is equivalent to the condition $w^{-1} \cdot \mu = \sum_{i=1}^k w_i^{-1} \cdot \mu_i$. Therefore by Theorem I condition (c) is equivalent to having a surjective cup product map

$$H^{\ell(w_1)}(X, L_{w_1^{-1} \cdot \mu_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{w_k^{-1} \cdot \mu_k}) \xrightarrow{\sqcup} H^{\ell(w)}(X, L_{w^{-1} \cdot \mu})$$

which, after dualizing gives an injective map

$$V_\mu \longrightarrow V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}.$$

In other words, we obtain a construction of V_μ as a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ realized through the cohomology of X .

Note that the conditions in 6.1.1 are *homogeneous*: If V_μ is a generalized PRV component, component of stable multiplicity one, or cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ then the same is true of $V_{m\mu}$ as a component of $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ for all $m \geq 1$. This follows immediately from the definitions.

6.2. Proof of Theorem II(a) and restatement of II(b).

Proof of Theorem II(a). Every cohomological component has multiplicity one by Theorem 3.9.1(b) (the condition on nonzero intersection holds by the nonsymmetric version of Lemma 2.6.1). By homogeneity we conclude that homological components are of stable multiplicity one. From Definition 6.1.1(a,c) it is clear that every cohomological component is a generalized PRV component. Thus every cohomological component is a generalized PRV component of stable multiplicity one. \square

For the proof of part (b) it will be more convenient to work with the symmetric form of the problem. Applying the symmetrization procedure from §2.8 (and replacing $k+1$ by k) we obtain the following reformulation of Theorem II(b).

Proposition (6.2.1) — Let μ_1, \dots, μ_k be dominant weights such that $\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k})^G = 1$ for $m \gg 0$ and suppose that we have elements w_1, \dots, w_k such that $\sum w_i^{-1}\mu_i = 0$. Then in either of the following two cases:

- (i) at least one of μ_1, \dots, μ_k is strictly dominant,
- (ii) G is a classical simple group or product of classical simple groups,

there exist $\bar{w}_1, \dots, \bar{w}_k \in \mathcal{W}$ such that

$$(6.2.2) \quad \sum_{i=1}^k \bar{w}_i^{-1}\mu_i = 0 \text{ and } \Delta^+ = \bigsqcup_{i=1}^k \Phi_{\bar{w}_i}.$$

The proof of Proposition 6.2.1 will be given in §6.8 after some preliminary reduction steps.

For the rest of this section we assume that we have fixed dominant weights μ_1, \dots, μ_k and Weyl group elements w_1, \dots, w_k satisfying the conditions of Proposition 6.2.1.

6.3. Outline of the proof of Proposition 6.2.1.

For $i = 1, \dots, k$ let P_i be the parabolic subgroup of G such that L_{μ_i} is the pullback to X of an ample line bundle $L_{\tilde{\mu}_i}$ on G/P_i . Set $M = G/P_1 \times \cdots \times G/P_k$ and $L = L_{\tilde{\mu}_1} \boxtimes \cdots \boxtimes L_{\tilde{\mu}_k}$. The

condition that $\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k})^G = 1$ for all $m \gg 1$ implies that the GIT quotient $M//G$ with respect to L is a point.

If w_1, \dots, w_k are elements such that $\sum_{i=1}^k w_i^{-1}\mu_i = 0$ then by Lemma 2.12.1, the point $q = (w_1^{-1}, \dots, w_k^{-1})$ is a semi-stable point of M with a closed orbit. Let $H \subseteq G$ be the stabilizer subgroup of q , and \mathcal{N}_q be the normal space to the orbit at q . By the Luna slice theorem and the fact that the GIT quotient $M//G$ is a point we conclude that $\text{Sym}^*(\mathcal{N}_q)^H$ is one-dimensional.

The explicit combinatorial formula for the weights appearing in \mathcal{N}_q shows that a necessary condition for a solution of (6.2.2) to exist is that there is $v \in \mathcal{W}$ such that the weights of $v\mathcal{N}_q$ are contained in Δ^- . In Proposition 6.6.1 below we formulate a condition which together with the necessary condition above guarantees the existence of a solution of (6.2.2). Together these two conditions are equivalent to the existence of a parabolic subalgebra \mathfrak{p} with reductive part $\text{Lie}(H)$ such that the weights of \mathcal{N}_q are contained in \mathfrak{p} .

Finally, we use the restriction that $\text{Sym}^*(\mathcal{N}_q)^H$ is one-dimensional to show the existence of such a parabolic subalgebra when G is a classical group, or for any semisimple group G under a genericity condition.

6.4. Stabilizer subgroup of a semi-stable T -fixed point.

Let P_i be the parabolic with roots $\Delta_{P_i} = \{\alpha \in \Delta \mid \kappa(\alpha, \mu_i) \geq 0\}$, and let $M_i = G/P_i$. The stabilizer subgroup of the point w_i^{-1} in M_i is $w_i^{-1}P_iw_i$, whose roots are

$$(6.4.1) \quad \Delta_{w_i^{-1}P_iw_i} = \left\{ \alpha \in \Delta \mid \kappa(w_i\alpha, \mu_i) \geq 0 \right\} = \left\{ \alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) \geq 0 \right\}.$$

Let $M = M_1 \times \cdots \times M_k$ and let q be the point $q = (w_1^{-1}, \dots, w_k^{-1})$ of M . We set $H = \bigcap_{i=1}^k w_i^{-1}P_iw_i$ to be the stabilizer subgroup of q . The condition $\sum_{i=1}^k w_i^{-1}\mu_i = 0$ in combination with (6.4.1) shows that the roots of H are given by

$$(6.4.2) \quad \Delta_H = \left\{ \alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) = 0 \text{ for } i = 1, \dots, k \right\}.$$

We conclude from (6.4.2) that H is a reductive subgroup of G . Noting that $T \subseteq H$, the following lemma is another immediate consequence of (6.4.2).

Lemma (6.4.3) — We have $H = T$ if and only if the span of $\{w_i^{-1}\mu_i\}_{i=1}^k$ intersects the interior of some Weyl chamber. This happens, for instance, if any one of the weights μ_i is strictly dominant.

6.5. Torus action at fixed points of M and combinatorial deductions.

Let $\mathcal{W}_i = \{w \in \mathcal{W} \mid w\mu_i = \mu_i\} \subseteq \mathcal{W}$ be the stabilizer subgroup of μ_i ; this is the Weyl group of P_i . We will need the formula for the formal character of the tangent space of M_i at a torus fixed point. Because of the way that the inverses of group elements enter into our formulas we make the following convention: For any element w of \mathcal{W} and any i we let

$w_{s(i)}$ and $w_{l(i)}$ be respectively the shortest and longest elements in the coset $\mathcal{W}_i w$. Recall also that for $\Phi \subseteq \Delta$, $\langle \Phi \rangle$ denotes the formal character of the tangent space of M_i at the torus fixed point corresponding to the coset $w_i^{-1} \mathcal{W}_i$.

With this convention, if w_i is any element of \mathcal{W} , the formal character of the tangent space of M_i at the torus fixed point corresponding to the coset $w_i^{-1} \mathcal{W}_i$ is

$$\text{Ch}(T_{w_i^{-1} M_i}) = \langle \Phi_{w_i, s(i)} \rangle + \langle -\Phi_{w_i, l(i)}^c \rangle = \left\langle \{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1} \mu_i) < 0\} \right\rangle.$$

The formal character of the tangent space of M at q is therefore

$$(6.5.1) \quad \text{Ch}(T_q M) = \sum_{i=1}^k \left(\langle \Phi_{w_i, s(i)} \rangle + \langle -\Phi_{w_i, l(i)}^c \rangle \right) = \sum_{i=1}^k \left\langle \{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1} \mu_i) < 0\} \right\rangle.$$

Note that the multiplicity of each root α in the equations above is the number of i for which $\kappa(\alpha, w_i^{-1} \mu_i) < 0$.

If $\alpha \notin \Delta_H$ then there is some i for which $\kappa(\alpha, w_i^{-1} \mu_i) \neq 0$ and hence, by the condition $\sum_{i=1}^k w_i^{-1} \mu_i = 0$, there is some i for which $\kappa(\alpha, w_i^{-1} \mu_i) < 0$, i.e., α must appear as a weight in $T_q M$. By looking at the positive roots of $T_q M$ we therefore conclude that

$$(6.5.2) \quad (\Delta^+ \setminus \Delta_H^+) = \bigcup \Phi_{w_i, s(i)}.$$

Let O_q be the G -orbit of q in M . Since H is the stabilizer of q , the formal character of the tangent space $T_q O_q$ is

$$(6.5.3) \quad \text{Ch}(T_q O_q) = \langle \Delta^+ \setminus \Delta_H^+ \rangle + \langle \Delta^- \setminus \Delta_H^- \rangle.$$

If $\mathcal{N}_q = T_q M / T_q O_q$ is the normal space to the orbit at q , then the union in (6.5.2) is disjoint if and only if the formal character of \mathcal{N}_q contains no positive root.

Let \mathcal{M} be the subspace of \mathfrak{g} spanned by the root spaces corresponding to the roots appearing in \mathcal{N}_q . Comparing the multiplicities in (6.5.1) and (6.5.3) we conclude that the roots of \mathcal{M} are

$$(6.5.4) \quad \Delta_{\mathcal{M}} = \left\{ \alpha \in \Delta \mid \kappa(\alpha, w_i^{-1} \mu_i) < 0 \text{ for at least two } i \in \{1, \dots, k\} \right\}.$$

Let $\mathfrak{s} = \text{Lie}(H)$; equations (6.5.4) and (6.4.2) show that \mathcal{M} is an \mathfrak{s} -submodule of \mathfrak{g} .

The point q is not the only torus fixed point in its orbit; for any $v \in \mathcal{W}$ we can act on the left to get the torus fixed point $vq = (vw_1^{-1}, \dots, vw_k^{-1})$. The weights of the normal space \mathcal{N}_{vq} to the G -orbit at vq are the result of acting on the weights of \mathcal{N}_q by v and are hence the roots appearing in $\text{Ch } v\mathcal{M}$.

Repeating the previous arguments with the new point vq and the new stabilizer group $vHv^{-1} = \text{Stab}(vq)$, gives the following result.

Lemma (6.5.5) — For any $v \in \mathcal{W}$ we have

$$(\Delta^+ \setminus \Delta_{vHv^{-1}}^+) = \bigsqcup_{i=1}^k \Phi_{(w_i v^{-1})_{s(i)}}$$

if and only if $v\mathcal{M} \subseteq \mathfrak{b}^-$.

6.6. Reduction to the existence of $\mathfrak{p}_{\mathcal{M}}$.

Proposition (6.6.1) — Suppose that there exists $v \in \mathcal{W}$ satisfying the conditions

- (i) $v\mathcal{M} \subseteq \mathfrak{b}^-$,
- (ii) there is an element $w \in \mathcal{W}$ such that $\Phi_w = \Delta_{vHv^{-1}}^+$.

Then there exist $\bar{w}_1, \dots, \bar{w}_k \in \mathcal{W}$ such that $\sum_{i=1}^k \bar{w}_i^{-1} \mu_i = 0$ and $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{\bar{w}_i}$.

Proof. By condition (i) and Lemma 6.5.5 we have $(\Delta^+ \setminus \Delta_{vHv^{-1}}^+) = \bigsqcup_{i=1}^k \Phi_{(w_i v^{-1})_{s(i)}}$. Set $\tilde{w}_{k+1} = w$, $\mu_{k+1} = 0$, and $\tilde{w}_i = (w_i v^{-1})_{s(i)}$ for $i = 1, \dots, k$. Conditions (i) and (ii) above and the original assumption about w_1, \dots, w_k imply

$$(6.6.2) \quad \sum_{i=1}^{k+1} \tilde{w}_i^{-1} \mu_i = 0 \text{ and } \Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{\tilde{w}_i}.$$

Equation (6.6.2) and Theorem I show that there is a surjective cup product map

$$H^{\ell(\tilde{w}_1)}(X, L_{\tilde{w}_1^{-1} \cdot \mu_1}) \otimes \cdots \otimes H^{\ell(\tilde{w}_{k+1})}(X, L_{\tilde{w}_{k+1}^{-1} \cdot \mu_{k+1}}) \xrightarrow{\cup} H^N(X, K_X).$$

Since $H^{\ell(\tilde{w}_{k+1})}(X, L_{\tilde{w}_{k+1}^{-1} \cdot \mu_{k+1}})$ is the trivial module, if we factor the map above by cupping the k -th and $(k+1)$ -st factors together first, we obtain a surjective cup product map onto $H^N(X, K_X)$ only involving the modules $V_{\mu_1}^*, \dots, V_{\mu_k}^*$. By invoking Theorem I again we conclude that there are $\bar{w}_1, \dots, \bar{w}_k$ such that

$$(6.6.3) \quad \sum_{i=1}^k \bar{w}_i^{-1} \mu_i = 0 \text{ and } \Delta^+ = \bigsqcup_{i=1}^k \Phi_{\bar{w}_i},$$

proving Proposition 6.6.1. □

Remark. If there do exist $\bar{w}_1, \dots, \bar{w}_k$ satisfying the conclusion of Proposition 6.2.1 it is not hard to show that there must exist $v \in \mathcal{W}$ so that (i) of 6.6.1 holds. As a consequence of our method of proof we see aposteriori that there must be a v so that both (i) and (ii) hold when G is a classical group or under a genericity condition. We do not know if condition (ii) is necessary in general.

It is useful to rephrase the conditions of Proposition 6.6.1 in terms of the existence of a particular parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$.

Lemma (6.6.4) — Let $\mathfrak{s} = \text{Lie}(H)$. Suppose that there exists a parabolic subalgebra \mathfrak{p}_M with reductive part \mathfrak{s} such that $M \subseteq \mathfrak{p}_M$. Then conditions (i) and (ii) of Proposition 6.6.1 hold.

Proof. Let \mathfrak{p}_M be such a parabolic subalgebra. Acting by an element $v \in W$ we can conjugate \mathfrak{p}_M so that $\mathfrak{b}^- \subseteq v\mathfrak{p}_M$. This implies that $vM \subseteq \mathfrak{b}^-$. Since $v\mathfrak{s}$ is the radical of a parabolic subalgebra containing \mathfrak{b}^- , if w is the longest element of the Weyl group of $v\mathfrak{s}$ then $\Phi_w = \Delta_{v\mathfrak{s}}^+ = \Delta_{vHv^{-1}}^+$. \square

Remark. If there exists $v \in W$ such that condition (ii) of Proposition 6.6.1 holds then one can show that $\mathfrak{p} := \mathfrak{b}^- + v\mathfrak{s}$ is a parabolic subalgebra of \mathfrak{g} . If condition (i) also holds for this v then $\mathfrak{p}_M := v^{-1}\mathfrak{p}$ is a parabolic subalgebra satisfying the conditions of Lemma 6.6.4. Therefore the existence of the parabolic \mathfrak{p}_M is equivalent to the conditions in Proposition 6.6.1. Since we will not need this direction of the equivalence we omit the justification of the first assertion.

6.7. GIT consequences of the stable multiplicity one condition.

Let L be the line bundle on M whose pullback to X^k is $L_{\mu_1} \boxtimes \cdots \boxtimes L_{\mu_k}$. Then L is a G -equivariant ample line bundle on M . By the stable multiplicity one condition we have $\dim(M, L^m)^G = 1$ for all $m \gg 1$, and so the GIT quotient $M//G$ is a point.

The weight of L at q is $\sum_{i=1}^k w_i^{-1} \mu_i = 0$. By Lemma 2.12.1 this means that q is a semi-stable point with a closed orbit. By the Luna slice theorem, [Lu1, Théorème du Slice Étale, pg. 97], $\text{Spec}(\text{Sym}(\mathcal{N}_q^*)^H)$ and the image of q in the GIT quotient $M//G$ have a common étale neighbourhood. Hence $\dim(\mathcal{N}_q/H) = \dim(M//G) = 0$, i.e., $\dim \text{Sym}^*(\mathcal{N}_q^*)^H = 1$. Passing to the level of Lie algebras and dualizing we obtain $\dim \text{Sym}^*(\mathcal{N}_q)^{\mathfrak{s}} = 1$.

Since M is isomorphic to an \mathfrak{s} -submodule of \mathcal{N}_q we arrive at the following consequence of the stable multiplicity one condition:

Lemma (6.7.1) — Under the hypotheses of Proposition 6.2.1 and with the notation of §6.5, we have $\dim \text{Sym}^*(M)^{\mathfrak{s}} = 1$, i.e., $\text{Sym}^*(M)^{\mathfrak{s}}$ consists of just the constants.

6.8. Proof of Proposition 6.2.1.

By Proposition 6.6.1 and Lemma 6.6.4, to prove Proposition 6.2.1 it is enough to show the existence of the parabolic subalgebra \mathfrak{p}_M . By Lemma 6.7.1 we may assume that $\dim \text{Sym}^*(M)^{\mathfrak{s}} = 1$.

Proof of 6.2.1(i) — If any one of the weights μ_1, \dots, μ_k is strictly dominant, or more generally, if the span of $\{w_i^{-1} \mu_i\}_{i=1}^k$ intersects the interior of some Weyl chamber then by Lemma 6.4.3 $H = T$ and so $\mathfrak{s} = \text{Lie}(T) = \mathfrak{t}$ and $\Delta_{\mathfrak{t}}^+ = \emptyset$. The condition that $\dim(\text{Sym}^*(M)^{\mathfrak{t}}) = 1$ is then equivalent to the condition that no non-trivial non-negative combination of weights of M is zero. Hence by Farkas's lemma the weights of M all lie strictly on one side of a hyperplane and the cone dual to the cone they span is open. We may therefore pick a weight in the interior of the dual cone which is not on any hyperplane of the

Weyl chambers. The roots lying on the positive side of this hyperplane give the parabolic subalgebra \mathfrak{p}_M . \square

Proof of 6.2.1(ii)— Equation (6.4.2) shows that the roots of \mathfrak{s} are given by the vanishing of linear forms and hence \mathfrak{s} is the reductive part of a parabolic subalgebra. Let \mathfrak{a} be the center of \mathfrak{s} . For any $\nu \in \mathfrak{a}^* \setminus \{0\}$ set

$$\mathfrak{g}^\nu = \left\{ x \in \mathfrak{g} \mid [t, x] = \nu(t)x \text{ for all } t \in \mathfrak{a} \right\}.$$

Following Kostant, [K3] we call $\nu \in \mathfrak{a}^* \setminus \{0\}$ an \mathfrak{a} -root if $\mathfrak{g}^\nu \neq 0$. Let \mathcal{R} be the set of \mathfrak{a} -roots of \mathfrak{g} and \mathcal{S} the subset of those \mathfrak{a} -roots appearing in M , so that $M = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^\nu$.

A subset \mathcal{R}' of \mathcal{R} is called *saturated* if whenever $\nu \in \mathcal{R}'$ and $r\nu \in \mathcal{R}$ for some $r \in \mathbf{Q}_+$ then $r\nu \in \mathcal{R}'$ as well. It follows from (6.5.4) that \mathcal{S} is a saturated subset of \mathcal{R} .

In the appendix we establish the following result (Theorem A.1).

Theorem—Let \mathfrak{g} be a classical Lie algebra, \mathfrak{s} be a subalgebra which is the reductive part of a parabolic subalgebra of \mathfrak{g} , \mathcal{S} be a saturated subset of the \mathfrak{a} -roots \mathcal{R} , and $M = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^\nu$. If $\dim(\mathrm{Sym}^*(M))^\mathfrak{s} = 1$, then there exists a parabolic subalgebra $\mathfrak{p}_M \subseteq \mathfrak{g}$ with reductive part \mathfrak{s} such that $M \subseteq \mathfrak{p}_M$.

Thus when G is a simple classical group or a product of simple classical groups, the above theorem along with the previous reductions establish Proposition 6.2.1. \square

6.9. Comments on the existence of \mathfrak{p}_M .

We want to prove the implication

$$(6.9.1) \quad \text{generalized PRV of stable multiplicity one} \implies \text{cohomological}$$

for components of a tensor product. Our strategy has been to reduce the proof of this implication to the existence of the parabolic \mathfrak{p}_M . As is clear from the proof of Proposition 6.2.1 we can establish (6.9.1) in a few more cases than stated in Theorem II(b). Namely we can replace (i) by the condition that the span of $\{w_i^{-1}\mu_i\}$ intersects the interior of some Weyl chamber, and in (ii) we can allow \mathfrak{g} to be a semisimple algebra whose simple ideals are classical or of type G_2 since Theorem A.1 extends to this case.

We also have a proof of the existence of \mathfrak{p}_M in the simply-laced case under a hypothesis somewhat complementary to the condition in (i): the semisimple part of each $w_i^{-1}P_i w_i$ be exactly H ; we omit this argument here.

Nonetheless, we have not been able to establish the implication (6.9.1) in full generality. There are two points where our strategy can be improved. First, it is not clear to us that condition (ii) in Proposition 6.6.1 is necessary, although we believe that it is, i.e., that proving the implication (6.9.1) is equivalent to showing the existence of \mathfrak{p}_M . Second, it is not clear to us how to fully utilize the condition that M comes from the data in the hypotheses of Proposition 6.2.1. One of the implications of this data is that $\dim \mathrm{Sym}^*(M)^\mathfrak{s} = 1$. In the classical cases (and G_2) or under the genericity condition our arguments show that for

any \mathfrak{s} -submodule \mathcal{M} of \mathfrak{g} satisfying the condition $\dim \text{Sym}^*(\mathcal{M})^{\mathfrak{s}} = 1$ there exists a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ as in Lemma 6.6.4. For groups of the other exceptional types we can find explicit examples of \mathfrak{s} -submodules $\mathcal{M} \subseteq \mathfrak{g}$ such that $\dim \text{Sym}^*(\mathcal{M})^{\mathfrak{s}} = 1$ but which are not contained in a parabolic $\mathfrak{p}_{\mathcal{M}}$ with reductive part \mathfrak{s} (see the Appendix). However we have not been able to turn these examples into counterexamples to (6.9.1) since we are not able to construct data satisfying the hypotheses in Proposition 6.2.1 which produce these \mathcal{M} 's. Perhaps a more careful analysis of the requirement that \mathcal{M} come from this particular kind of representation-theoretic problem would be sufficient to prove (6.9.1) in general.

7. LITTLEWOOD-RICHARDSON CONE AND BOUNDS ON MULTIPLICITIES

In this section we relate cohomological components to the boundary of codimension n of the Littlewood-Richardson cone, where n is the rank of G . We extend the argument used in Lemma 3.8.1(c) to give a bound on the multiplicities of components generalizing that of Theorem 3.9.1(b). We show how this bound imposes conditions on the order of growth of multiplicities on the boundary of the cone.

We are working over an algebraically closed field of characteristic zero. For notational convenience we will assume that the field is C . It is advisable to look over §2.7 before reading this section.

7.1. Cohomological components are of codimension n .

Proposition (7.1.1) — Let μ_1, \dots, μ_k , and μ be dominant weights. If V_{μ} is a cohomological component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$ then μ is a vertex of the Littlewood-Richardson polytope $\mathcal{P}(\mu_1, \dots, \mu_k)$.

Proof. Recall that for any polytope \mathcal{P} in a vector space E every point $p \in \mathcal{P}$ lies on the relative interior of a unique face. The dimension of this face is the same as the dimension of the subspace of E spanned by the set $\{\epsilon \in E \mid p \pm \epsilon \in \mathcal{P}\}$.

Let w_1, \dots, w_k , and w be the elements from §6.1.1(c) expressing V_{μ} as a cohomological component of $V_{\mu_1} \otimes \dots \otimes V_{\mu_k}$. If ϵ is such that $(\mu_1, \dots, \mu_k, \mu \pm \epsilon)$ is in $\mathcal{P}(\mu_1, \dots, \mu_k)$ we must have that $\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1}(\mu \pm \epsilon) = \mp w^{-1} \epsilon$ belongs to $\text{span}_{Q \geq 0} \Delta^+$. This is only possible if $w^{-1} \epsilon = 0$, i.e., $\epsilon = 0$ and so μ is a vertex of $\mathcal{P}(\mu_1, \dots, \mu_k)$. \square

Lemma (7.1.2) — With the notation of §2.7, if $I = \emptyset$ then w_1, \dots, w_k , and w satisfy (2.7.1) with respect to I if and only if $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ and $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$.

Proof. If $I = \emptyset$ then (2.7.1)(iii) becomes $\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 = 0$. Hence by (2.7.1)(i) and Theorem 3.9.1(d) we conclude that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$. Conversely, if w_1, \dots, w_k , and w are such that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ and $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$ then (2.7.1)(i,iii) hold and (2.7.1)(ii) is automatic since $P_I = B$ and so $\mathcal{W}_{P_I} = \{e\}$. \square

Theorem IV now follows immediately by combining Theorem 2.7.2, Lemma 7.1.2, and Theorem I.

Lemma (7.1.3) — Suppose w_1, \dots, w_k , and w are such that $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ and $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = 1$. Then for any set I of simple roots, if we set \bar{w}_i to be the shortest element in $w_i \mathcal{W}_{P_I}$ for $i = 1, \dots, k$, and \bar{w} to be the shortest element in $w \mathcal{W}_{P_I}$ then $\bar{w}_1, \dots, \bar{w}_k$ and \bar{w} satisfy (2.7.1) with respect to I .

Proof. The only part of (2.7.1) which is not immediate is that $\cap_{i=1}^k [\Omega_{\bar{w}_i}] \cdot [X_{\bar{w}}] = 1$. Since we will only use this lemma to derive a picture of the Littlewood-Richardson cone around a cohomological component in Corollary 7.3.5(b), we only sketch the argument.

Let G' be the Levi component of P_I , $B' = G' \cap B$ and $X' = G'/B'$. Write $w_i = \bar{w}_i u_i$ for $i = 1, \dots, k$, and $w = \bar{w} u$ with each u_i , respectively u in \mathcal{W}_{P_I} . Let $\pi: X \rightarrow G/P_I$ be the projection. The general fibre of $\pi|_{\Omega_{w_i}}: \Omega_{w_i} \rightarrow \pi(\Omega_{w_i})$ has cohomology class $[\Omega'_{u_i}]$ in $H^*(X', \mathbb{Z})$ and similarly the general fibre of $\pi|_{X_w}: X_w \rightarrow \pi(X_w)$ has cohomology class $[X'_u]$. Furthermore, the intersection numbers $\cap_{i=1}^k [\Omega_{\bar{w}_i}] \cdot [X_{\bar{w}}]$ and $\cap_{i=1}^k [\pi(\Omega_{w_i})] \cdot [\pi(X_w)]$ are equal, the last intersection being computed in $H^*(G/P_I, \mathbb{Z})$. Using Kleiman's transversality theorem one can factor the intersection number on X as a product of an intersection number on G/P_I and an intersection number on the fibre:

$$(7.1.4) \quad \cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] = \left(\cap_{i=1}^k [\Omega_{\bar{w}_i}] \cdot [X_{\bar{w}}] \right) \left(\cap_{i=1}^k [\Omega'_{u_i}] \cdot [X'_u] \right).$$

By hypothesis the left side of (7.1.4) is equal to one. Therefore both of the intersection numbers on the right side are equal to one, and in particular $\cap_{i=1}^k [\Omega_{\bar{w}_i}] \cdot [X_{\bar{w}}] = 1$. \square

Note that the evaluation of the condition in Theorem 2.7.2(a) does not depend on the representatives chosen in the cosets $w_i \mathcal{W}_{P_I}$. Thus by Lemma 7.1.3 if one considers the conditions imposed on $(\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k)$ obtained by writing $\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu$ in terms of the simple roots and requiring that any subset of size r of the coordinates vanish, one obtains a codimension $n - r$ face of $\mathcal{C}(k)$. Each of these faces contain the codimension n face where all coefficients are zero, consisting of cohomological components.

7.2. Bounds on multiplicities. Recall that $\text{mult}(V_\mu, V)$ denotes the multiplicity of V_μ in V .

Definition (7.2.1) — For any multiset \mathcal{S} with support in Δ^+ (i.e., \mathcal{S} is a set of positive roots with multiplicities) we define the partition function $p_{\mathcal{S}}$ by setting $p_{\mathcal{S}}(\gamma)$ to be the coefficient of e^γ in the series $\prod_{\alpha \in \Delta^+} \frac{1}{(1-e^\alpha)^{m_{\mathcal{S}}(\alpha)}}$, where $m_{\mathcal{S}}(\alpha)$ is the multiplicity of α in \mathcal{S} .

If $\mathcal{S} = \Delta^+$ then $p_{\mathcal{S}}$ is the usual Kostant partition function. For any \mathcal{S} , $p_{\mathcal{S}}(0) = 1$, and $p_{\mathcal{S}}(\gamma) = 0$ if $\gamma \notin \text{span}_{\mathbb{Z} \geq 0} \mathcal{S}$ (in particular, $p_{\mathcal{S}}(\gamma) = 0$ if $\gamma \notin \text{span}_{\mathbb{Z} \geq 0} \Delta^+$).

By convention the multiplicities in a multiset are always non-negative. If \mathcal{S}' is a multiset with support in Δ^+ , then the set difference $\mathcal{S}' \setminus \Delta^+$ is the multiset obtained by reducing

each of the nonzero multiplicities in \mathcal{S}' by one. The operation of union with multiplicity consists of adding the multiplicities and is denoted by \bigcup^{mult} .

Theorem (7.2.2) — Let w_1, \dots, w_k , and w be such that $\cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w] \neq 0$ and set

$$\mathcal{S} = \left(\left(\bigcup_{i=1}^k \Phi_{w_i}^c \right) \bigcup^{\text{mult}} \Phi_w \right) \setminus \Delta^+.$$

Then for any dominant weights μ_1, \dots, μ_k , and μ we have the bound

$$\text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}) \leq p_{\mathcal{S}} \left(\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu \right).$$

Proof. As in the proof of Theorem 3.9.1 let $v_i = w_i^{-1} w_0$ for $i = 1, \dots, k$, $v_{k+1} = w^{-1}$, let \underline{v}_i be a reduced word with product v_i for $i = 1, \dots, k+1$, and set $\underline{v} = (\underline{v}_1, \dots, \underline{v}_{k+1})$. By Corollary 3.7.5 the degree of $f_{\underline{v}}: Y_{\underline{v}} \rightarrow X^{k+1}$ is given by the intersection number $\cap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \cap_{i=1}^k [\Omega_{w_i}] \cdot [X_w]$ which is nonzero by assumption, i.e., $f_{\underline{v}}$ is surjective.

Let $\lambda_i = -w_0 \mu_i$ for $i = 1, \dots, k$ and $\lambda_{k+1} = \mu$. Set L to be the line bundle $L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_{k+1}}$ on X^{k+1} , so that the multiplicity of V_μ in $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is equal to $\dim H^0(X^{k+1}, L)^G$. Since $f_{\underline{v}}$ is surjective, pullback induces an inclusion $H^0(Y_{\underline{v}}, f_{\underline{v}}^* L)^G \xhookrightarrow{f_{\underline{v}}^*} H^0(X^{k+1}, L)^G$.

As in the proof of Lemma 3.8.1, let $Z := Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ be the fibre of f_{\circ} over $e \in X$. The pushforward $f_{\circ*}(f_{\underline{v}}^* L)$ is a vector bundle on X whose fibre over e is $H^0(Z, f_{\underline{v}}^* L|_Z)$. By the equivalence between G -invariant sections of a G -bundle on X and B -invariant vectors of the fibre over e , restriction to Z induces an isomorphism

$$H^0(Z, f_{\underline{v}}^* L|_Z)^B \xrightarrow{\sim} H^0(Y_{\underline{v}}, f_{\underline{v}}^* L)^G.$$

Let $p = p_{\underline{v}} \in Z$ be the maximum point of $Y_{\underline{v}}$. Either by the theorem of Bialynicki-Birula [Bial] or by following through the construction in §2.11 one checks that there is a T -stable open affine subset $U \subset Z$ containing p and isomorphic to affine kN -space. In fact

$$(7.2.3) \quad U = (Bv_1 B)/B \times (Bv_2 B)/B \times \cdots \times (Bv_{k+1} B)/B$$

and so U is B -stable. Since U is open in the irreducible variety Z , restriction gives an inclusion $H^0(U, f_{\underline{v}}^* L|_U)^B \hookrightarrow H^0(Z, f_{\underline{v}}^* L|_Z)^B$. Thus to bound the $\text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})$ it suffices to bound $\dim H^0(U, f_{\underline{v}}^* L|_U)^B$. Set $\gamma = \sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu$; we will show that $\dim H^0(U, f_{\underline{v}}^* L|_U)^B \leq p_{\mathcal{S}}(\gamma)$.

Let $L_U = (f_{\underline{v}}^* L)|_U$. Since U is isomorphic to affine space, L_U is (non-equivariantly) trivial on U . Let s_{\circ} be a section of L_U which is nowhere vanishing. The torus T takes s_{\circ} to another nowhere vanishing section which must therefore be a multiple of s_{\circ} , i.e., T acts on s_{\circ} via a weight. This must be the same as the weight of the action of L_U at p , and so T acts on s_{\circ} with weight γ . Let B^+ be the unipotent radical of B . By the same reasoning, B^+ must take

s_\circ to a multiple of itself. Since B^+ has only the trivial one-dimensional representation s_\circ must be fixed by B^+ .

Every section $s \in H^0(U, L_U)$ can be written as $s = s_\circ h$ for some function $h \in H^0(U, \mathcal{O}_U)$. The section s is B -invariant if and only if h is B^+ -invariant and h is an eigenfunction of T on which T acts via $-\gamma$. For any weight δ , let $H^0(U, \mathcal{O}_U)_\delta$ denote the space of eigenfunctions of T on which T acts via δ . Let \mathfrak{n} be the Lie algebra of B^+ (i.e., the nilpotent radical of \mathfrak{b}); \mathfrak{n} acts on $H^0(U, \mathcal{O}_U)$ via derivations. By the correspondence above between sections of L_U and functions on U we have $\dim H^0(U, L_U)^B = \dim H^0(U, \mathcal{O}_U)_{-\gamma}^\mathfrak{n}$.

Let $U_i = (Bv_i B)/B$. The formal character of the tangent space $T_{p_{v_i}} U_i = T_{p_{v_i}} Z_{v_i}$ is $\langle \Phi_{v_i^{-1}} \rangle$ (see (2.11.3)) and hence the affine space U_i is equal to $\text{Spec}(\mathbf{C}[z_{i,-\alpha}]_{\alpha \in \Phi_{v_i^{-1}}})$ where each $z_{i,-\alpha}$ is an independent variable on which T acts via the weight $-\alpha$. Since the roots in $\Phi_{v_i^{-1}}$ are distinct, the variables $z_{i,-\alpha}$ are uniquely determined up to scalar.

Let $R = \mathbf{C}[z_{i,-\alpha}]_{i=1,\dots,k+1,\alpha \in \Phi_{v_i^{-1}}}$. By (7.2.3) and the description of U_i above $R = H^0(U, \mathcal{O}_U)$ (or, equivalently, $U = \text{Spec}(R)$). Let

$$\mathcal{S}' = \bigcup_{i=1}^{\text{mult } k+1} \Phi_{v_i^{-1}} = \left(\bigcup_{i=1}^{\text{mult } k} \Phi_{w_i}^c \right) \bigcup^{\text{mult }} \Phi_w$$

so that $\mathcal{S} = \mathcal{S}' \setminus \Delta^+$. The number of monomials in the variables $\{z_{i,-\alpha}\}$ of weight $-\gamma$ is $p_{\mathcal{S}'}(\gamma)$, i.e., $\dim H^0(U, \mathcal{O}_U)_{-\gamma} = p_{\mathcal{S}'}(\gamma)$. By analyzing the condition of \mathfrak{n} -invariance we will obtain the bound $\dim H^0(U, \mathcal{O}_U)_{-\gamma}^\mathfrak{n} \leq p_{\mathcal{S}}(\gamma)$.

For each $\beta \in \Delta^+$ let ∂_β be the vector field giving the action of a nonzero element of $\mathfrak{g}^\beta \subseteq \mathfrak{n}$ on U . Each ∂_β is a first-order differential operator with polynomial coefficients. Separating the coefficients into homogeneous pieces we may write

$$\partial_\beta = \partial_\beta^0 + \partial_\beta^1 + \partial_\beta^2 + \cdots$$

where ∂_β^j denotes the component of ∂_β whose coefficients are homogeneous of degree j . In particular,

$$\partial_\beta^0 = \sum_{i: \beta \in \Phi_{v_i}^{-1}} c_{(i,\beta)} \partial_{z_{i,-\beta}}$$

where $c_{(i,\beta)} \in \mathbf{C}$ are nonzero constants. Each ∂_β is an operator of Δ -degree β , i.e.,

$$\partial_\beta (H^0(U, \mathcal{O}_U)_\delta) \subseteq H^0(U, \mathcal{O}_U)_{\delta+\beta}$$

for each weight δ . Thus, setting $\partial_1 = \sum_{\beta \in \Delta^+} \partial_\beta$ we obtain

$$\begin{aligned} H^0(U, \mathcal{O}_U)_{-\gamma}^\mathfrak{n} &= \bigcap_{\beta \in \Delta^+} \ker \left(H^0(U, \mathcal{O}_U)_{-\gamma} \xrightarrow{\partial_\beta} H^0(U, \mathcal{O}_U)_{-\gamma+\beta} \right) \\ &= \ker \left(H^0(U, \mathcal{O}_U)_{-\gamma} \xrightarrow{\partial_1} \bigoplus_{\beta \in \Delta^+} H^0(U, \mathcal{O}_U)_{-\gamma+\beta} \right). \end{aligned}$$

In order to estimate the size of the kernel we deform the differential operator. For any $s \in \mathbf{C}$ and $\beta \in \Delta^+$ let

$$\partial_{\beta,s} = \partial_\beta^0 + s\partial_\beta^1 + s^2\partial_\beta^2 + \dots.$$

Each $\partial_{\beta,s}$ is again a homogeneous operator of Δ -degree β . Set $\partial_s = \sum_{\beta \in \Delta^+} \partial_{\beta,s}$ and consider the maps

$$(7.2.4) \quad H^0(U, \mathcal{O}_U)_{-\gamma} \xrightarrow{\partial_s} \bigoplus_{\beta \in \Delta^+} H^0(U, \mathcal{O}_U)_{-\gamma+\beta}.$$

If $s \neq 0$ then by simultaneously scaling the variables $z_{i,-\alpha}$ we can transform ∂_s into a nonzero multiple of ∂_1 , i.e., for all $s \neq 0$ the dimension of the kernel of ∂_s is the same. Specializing to $s = 0$ can only cause the kernel to increase and therefore $\dim H^0(U, \mathcal{O}_U)_{-\gamma}^n = \dim \ker(\partial_1) \leq \dim \ker(\partial_0)$.

Each $\partial_{\beta,0} = \partial_\beta^0$ is a constant coefficient differential operator. Therefore if R_1 is the degree one piece of $H^0(U, \mathcal{O}_U)$ and R_1° the subspace annihilated by $\{\partial_{\beta,0}\}_{\beta \in \Delta^+}$ then the subring of $H^0(U, \mathcal{O}_U)$ consisting of polynomials annihilated by $\{\partial_{\beta,0}\}_{\beta \in \Delta^+}$ is $\text{Sym}^\cdot(R_1^\circ)$.

For any weight δ let $R_{1,\delta}$ and $R_{1,\delta}^\circ$ denote the subspaces of weight δ of R_1 and R_1° respectively. Since $\partial_{\beta,0}$ acts only on the variables of weight $-\beta$, if $R_{1,-\beta} \neq 0$ then $R_{1,-\beta}^\circ$ is of codimension one in $R_{1,-\beta}$. I.e., the effect of each $\partial_{\beta,0}$ is to “reduce the number of variables of weight $-\beta$ by one”. Therefore $\dim \ker \partial_0 = \dim \text{Sym}^\cdot(R_1^\circ)_{-\gamma} = p_S(\gamma)$. \square

7.3. Corollaries of Theorem 7.2.2. Note that if $\Phi_w = \sqcup_{i=1}^k \Phi_{w_i}$ then $\mathcal{S} = (k-1)\Delta^+$, i.e, the multiset in which each positive root has multiplicity $k-1$. Applying Theorem 7.2.2 yields

$$(7.3.1) \quad \text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}) \leq p_{(k-1)\Delta^+} \left(\sum_{i=1}^k w_i^{-1} \mu_i - w^{-1} \mu \right).$$

The most interesting case is when $k = 2$, i.e. $\mathcal{S} = \Delta^+$. Taking $w_1 = w_2 = w = e$ in (7.3.1) we obtain

Corollary (7.3.2) — If μ_1, μ_2 , and μ are dominant weights, then

$$(7.3.3) \quad \text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) \leq p(\mu_1 + \mu_2 - \mu),$$

where $p = p_{\Delta^+}$ is the usual Kostant partition function.

Remark. The right hand side of (7.3.3) is the leading term of Steinberg’s multiplicity formula [St]. Since Steinberg’s formula is a signed double summation over the Weyl group, this bound does not seem to be obvious.

Remark. If μ is sufficiently close to $\mu_1 + \mu_2$ then the Brauer-Klimyk multiplicity formula shows that $\text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) = p(\mu_1 + \mu_2 - \mu)$, i.e., for μ sufficiently close to $\mu_1 + \mu_2$, the upper bound in Corollary 7.3.2 is exact.

For any set I of simple roots, we define Δ_I to be the roots of the Levi component of P_I . Equivalently, Δ_I is the subset of Δ consisting of those roots in $\text{span}_{\mathbb{Z}} I$.

Corollary (7.3.4) — Suppose that $(\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k)$ is on a face of the type from Theorem 2.7.2(a). Let I be the corresponding set of simple roots, and w_1, \dots, w_k , and w be the corresponding elements satisfying conditions (2.7.1) with respect to I . Set $\gamma = \sum_{i=1}^k w_i^{-1} \mu_k - w^{-1} \mu$. Then

$$\text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}) \leq p_{(k-1)\Delta_I^+}(\gamma).$$

Proof. For any set I of simple roots and any element $v \in \mathcal{W}$ the shortest element in $v\mathcal{W}_{P_I}$ is the unique element $v' \in v\mathcal{W}_{P_I}$ such that $\Phi_{v'} \cap \Delta_I^+ = \emptyset$. Applying this observation to the elements w_1, \dots, w_k and w , we see that

$$S = \left(\left(\bigcup_{i=1}^k \Phi_{w_i}^c \right) \bigcup^{\text{mult}} \Phi_w \right) \setminus \Delta^+ = (k-1)\Delta_I^+ \bigsqcup^{\text{mult}} \left\{ \begin{array}{l} \text{roots outside of } \Delta_I^+ \text{ counted} \\ \text{with some multiplicities} \end{array} \right\}.$$

By Theorem 7.2.2 we know that $p_S(\gamma)$ bounds $\text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k})$. To write a root outside of Δ_I^+ as a combination of simple roots it is necessary to use at least one simple root α_j not contained in I . By the condition of Theorem 2.7.2(a) if we write γ as a combination of simple roots the coefficient of every simple root α_j outside of I is zero. Therefore the roots outside of Δ_I^+ do not contribute to $p_S(\gamma)$ and so $p_S(\gamma) = p_{(k-1)\Delta_I^+}(\gamma)$. \square

Corollary (7.3.5) —

- (a) Suppose that F is a face of $\mathcal{C}(2)$ of codimension $n-1$ which intersects the locus of strictly dominant weights. Then for every triple (μ_1, μ_2, μ) of weights in F , $\text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) \leq 1$. After scaling (i.e., modulo the saturation problem) $\text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) = 1$.
- (b) If G is classical, then for every codimension n face F' of cohomological components, there is a face F of codimension $\lfloor \frac{n}{2} \rfloor$ containing F' such that for every triple (μ_1, μ_2, μ) of weights in F , $\text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) \leq 1$ (and after scaling, $\text{mult}(V_\mu, V_{\mu_1} \otimes V_{\mu_2}) = 1$).

Proof. If F is a codimension $n-1$ face which intersects the locus of strictly dominant weights then by Theorem 2.7.2(b) F is of the type in Theorem 2.7.2(a) with I a single simple root. Since the partition function $p_{\Delta_I^+}$ is the constant function 1 for every γ in the span of I , part (a) follows from Corollary 7.3.4.

For (b), pick I to be the largest set of simple roots such that the subdiagram of the Dynkin diagram of G corresponding to I consists of disconnected A_1 pieces. Then the partition function $p_{\Delta_I^+}$ is also 1 for every γ in the span of Δ_I^+ .

Let w_1, w_2 , and w be the elements defining F' , and \bar{w}_1, \bar{w}_2 , and \bar{w} the elements of shortest length in the corresponding left cosets of \mathcal{W}_{P_I} . By Lemma 7.1.3, \bar{w}_1, \bar{w}_2 , and \bar{w} satisfy (2.7.1) with respect to I . Let F be the face defined by this data. Corollary 7.3.4 then gives the multiplicity bound 1. \square

Remark. In Corollary 7.3.5 one can do slightly better when G is of type D_n : there is a face F of codimension $\lfloor \frac{n-1}{2} \rfloor$ with this property. For the exceptional Lie algebras, the proof above guarantees a face of codimension 1, 2, 3, 4, and 4, if G is isomorphic to G_2 , F_4 , E_6 , E_7 , and E_8 respectively.

Remark. After Corollary 7.3.2 we noted that near the highest weight $\mu_1 + \mu_2$ the bound given there is exact. If $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ and $([\Omega_{w_1}] \cap [\Omega_{w_2}]) \cdot [X_w] = 1$ then experimental evidence suggests that the bound in (7.3.1) is also exact when μ is sufficiently close to the cohomological face determined by w_1 , w_2 , and w . In other words, that the multiplicities near a cohomological component are given by the Kostant partition function applied to the sum $w_1^{-1}\mu_1 + w_2^{-1}\mu_2 - w^{-1}\mu$ measuring the distance between μ and that component. Corollary 7.3.5 above establishes some consequences consistent with this description, but without assuming it. We illustrate this behaviour with the following example.

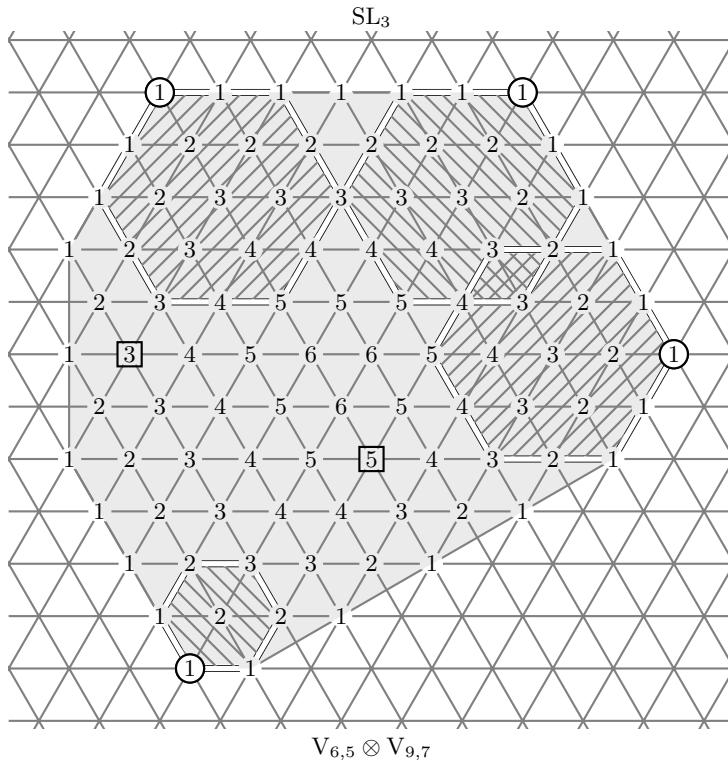


Figure 2

In the figure above we have outlined hexagons at each cohomological component where the multiplicities behave as proscribed by the Kostant partition function.

7.4. Order of growth of partition functions. Suppose that we are given a matrix A with integer entries. The *vector partition function* corresponding to A counts, for each γ in the span of the columns of A , the number of vectors v with non-negative integer entries such that $Av = \gamma$.

Both the Kostant partition function and its extension to multisets above are examples of vector partition functions. For the partition function p_S one takes the matrix A whose

rows are indexed by the simple roots and whose columns are indexed by the elements of \mathcal{S} (repeated according to their multiplicities). The column corresponding to an entry of \mathcal{S} is its expression in terms of the simple roots.

A result of Blakley [Bl] (see also [Stu, Theorem 1]) shows that for any vector partition function one can divide the space spanned by the columns of A into chambers so that the partition function is quasi-polynomial on each chamber of degree at most $\#(\text{columns of } A) - \text{rank } A$. When $\mathcal{S} = (k-1)\Delta_I^+$ this structure theorem for partition functions combined with Corollary 7.3.4 gives the following result.

Corollary (7.4.1) — Suppose that $(\mu_1, \dots, \mu_k, \mu) \in \mathcal{C}(k)$ is on a face of the type from Theorem 2.7.2(a). Then the function

$$(7.4.2) \quad m \mapsto \text{mult}(V_{m\mu}, V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k})$$

is bounded above by a polynomial in m of degree $(k-1)\#\Delta_I^+ - \#I$.

Remark. The function (7.4.2) can also be viewed as the Hilbert function of an appropriate GIT quotient of X^{k+1} (or some product of G/P 's if not every weight is strictly dominant). Corollary 7.4.1 can therefore also be viewed as a bound on the dimension of the GIT quotients when the line bundle giving the GIT quotient comes from a face of $\mathcal{C}(k)$.

Remark. Heckman established [He, (3.17)] that the asymptotic growth of the multiplicity function

$$(\mu, \mu_1, \dots, \mu_k) \mapsto \text{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}),$$

when viewed as a function on Λ_I^{k+1} , is polynomial of degree $(k-1)\#\Delta^+ - n$. Equation (7.3.1) provides a family of bounds (one for each w_1, \dots, w_k , and w satisfying (1.2.1)) each of which has the same order of growth as the actual multiplicity function. It would be interesting to compare the leading terms given by (7.3.1) with the ones in [He].

A. APPENDIX ON INVARIANTS OF $\text{Sym}^\cdot \mathfrak{g}$

Let \mathfrak{s} be a subalgebra of \mathfrak{g} which is the reductive part of a parabolic subalgebra of \mathfrak{g} and let \mathfrak{a} be the center of \mathfrak{s} . Following Kostant, [K3], for any $\nu \in \mathfrak{a}^* \setminus \{0\}$ we set

$$\mathfrak{g}^\nu = \left\{ x \in \mathfrak{g} \mid [t, x] = \nu(t)x \text{ for all } t \in \mathfrak{a} \right\}.$$

An element $\nu \in \mathfrak{a}^* \setminus \{0\}$ is called an \mathfrak{a} -root if $\mathfrak{g}^\nu \neq 0$. The properties of \mathfrak{a} -roots and the corresponding \mathfrak{a} -root decomposition of \mathfrak{g} inherit most of the properties of the usual roots and the usual root decomposition of \mathfrak{g} , as established by Kostant. We refer the reader to [K3] for a complete account of the theory of \mathfrak{a} -roots.

Let \mathcal{R} denote the set of \mathfrak{a} -roots. A subset \mathcal{S} of \mathcal{R} is called *saturated* if whenever $\nu \in \mathcal{S}$ and $r\nu \in \mathcal{R}$ for some $r \in \mathbf{Q}_+$ then $r\nu \in \mathcal{S}$ as well. With this notation we prove:

Theorem (A.1) — Let \mathfrak{g} be a simple classical Lie algebra, \mathfrak{s} be a subalgebra which is the reductive part of a parabolic subalgebra of \mathfrak{g} , \mathfrak{a} be the center of \mathfrak{s} , \mathcal{S} be a saturated subset

of the \mathfrak{a} -roots of \mathfrak{g} , and $\mathcal{M} = \bigoplus_{\nu \in S} \mathfrak{g}^\nu$. If $\dim(\text{Sym}^r(\mathcal{M}))^{\mathfrak{s}} = 1$, then there exists a parabolic subalgebra $\mathfrak{p}_\mathcal{M}$ of \mathfrak{g} with reductive part \mathfrak{s} such that $\mathcal{M} \subset \mathfrak{p}_\mathcal{M}$.

Theorem A.1 also holds if $\mathfrak{g} = G_2$; we will give examples to show that it does not hold for the other exceptional algebras.

First we describe the parabolic subalgebras and the corresponding sets \mathcal{R} for the classical Lie algebras. For convenience of notation we will work with the reductive Lie algebra gl_n instead of sl_n . A partition $\mathcal{P} = \{I_1, \dots, I_k\}$ of $\{1, \dots, n\}$ is *linearly ordered* if the set of its parts is linearly ordered. We write $\mathcal{P}(i)$ for the part of \mathcal{P} which contains i . The inequalities $\mathcal{P}(i) \prec \mathcal{P}(j)$ and $\mathcal{P}(i) \preceq \mathcal{P}(j)$ are taken in the linear order of the parts of \mathcal{P} . For the standard basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of \mathfrak{t}^* we denote the dual basis of \mathfrak{t} by $\{h_1, \dots, h_n\}$. A linear order on the set $\{\pm\delta_1, \dots, \pm\delta_k\}$ is *compatible with multiplication by -1* if $\pm\delta_i \prec \pm\delta_j$ implies $\mp\delta_j \prec \mp\delta_i$. To simplify notation we adopt the convention that B_1 , respectively C_1 , is a subalgebra of $\mathfrak{g} = B_n$, respectively $\mathfrak{g} = C_n$, isomorphic to A_1 and whose roots are short, respectively long roots, of \mathfrak{g} . The subalgebras $D_2 = A_1 \oplus A_1$ and $D_3 = A_3$ of D_n have similar meaning.

Let \mathfrak{g} be of type $X_n = A_n, B_n, C_n$, or D_n and let \mathfrak{s} be a subalgebra of \mathfrak{g} which is the reductive part of a parabolic subalgebra of \mathfrak{g} . Every simple ideal of \mathfrak{s} is isomorphic to A_r or X_r for some r . Furthermore, if \mathfrak{g} is not of type A_n , \mathfrak{s} has at most one simple ideal of type X_r . For \mathfrak{g} of type $X_n = B_n, C_n$, or D_n the parabolic subalgebras of \mathfrak{g} are split into two types depending on whether their reductive parts contain (Type II) or do not contain (Type I) a simple ideal of type X_r (including B_1, C_1, D_2 , or D_3).

In the description of the combinatorics of the simple classical Lie algebras below, the formulas for their parabolic subalgebras \mathfrak{p} containing a fixed reductive part \mathfrak{s} look very uniform (e.g. 11). In some instances this is misleading since the formulas do not explicitly indicate the subalgebra \mathfrak{s} which, however, is an integral part of the structure of \mathfrak{p} .

Roots and parabolic subsets of the classical Lie algebras. We now list the combinatorial descriptions of the parabolic subalgebras and related data in the classical cases.

$\mathfrak{g} = A_n$

1. The roots of \mathfrak{g} are: $\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$
2. Parabolic subalgebras of \mathfrak{g} are in one-to-one correspondence with:

linearly ordered partitions $\mathcal{P} = (I_1, \dots, I_k)$ of $\{1, \dots, n\}$.

Given a linearly ordered partition \mathcal{P} ,

3. The roots of $\mathfrak{p}_{\mathcal{P}}$ are $\{\varepsilon_i - \varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\}$
4. The roots of $\mathfrak{s}_{\mathcal{P}}$ are $\{\varepsilon_i - \varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$
5. $\mathfrak{s}_{\mathcal{P}} = \bigoplus_i \mathfrak{s}_{\mathcal{P}}^i$, where $\mathfrak{s}_{\mathcal{P}}^i \cong gl_{|I_i|}$;
6. The Cartan subalgebra of $\mathfrak{s}_{\mathcal{P}}^i$ is spanned by $\{h_j\}_{j \in I_i}$
7. The roots of $\mathfrak{s}_{\mathcal{P}}^i$ are $\{\varepsilon_j - \varepsilon_l \mid j \neq l \in I_i\}$.
8. $\mathfrak{a}_{\mathcal{P}}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|I_i|} \sum_{j \in I_i} h_j$

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\delta_i - \delta_j \mid 1 \leq i \neq j \leq k\}$.

10. For $\nu = \delta_i - \delta_j \in \mathcal{R}$, $\mathfrak{s}_{\mathcal{P}}$ -module $\mathfrak{g}^\nu \cong V_i \otimes V_j^*$, where V_i and V_j^* are the natural $\mathfrak{s}_{\mathcal{P}}^i$ -module and the dual of the natural $\mathfrak{s}_{\mathcal{P}}^j$ -module respectively, all other factors of $\mathfrak{s}_{\mathcal{P}}$ acting trivially.
11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P}}$ are in a bijection with the ordered partitions \mathcal{Q} of $\{1, \dots, n\}$ whose parts are the same as the parts of \mathcal{P} or, equivalently, with linear orders on the set $\{\delta_1, \dots, \delta_k\}$.

$\mathfrak{g} = B_n$

1. The roots of \mathfrak{g} are: $\Delta = \{\pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i \mid 1 \leq i \neq j \leq n\}$.
2. Parabolic subalgebras of \mathfrak{g} are in one-to-one correspondence with:

Type I: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ and $\sigma: \{1, \dots, n\} \rightarrow \{\pm 1\}$ is a choice of signs.

Type II: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_0, I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ with largest element I_0 and $\sigma: \{1, \dots, n\} \setminus I_0 \rightarrow \{\pm 1\}$ is a choice of signs.

In Type I:

3. The roots of $\mathfrak{p}_{(\mathcal{P}, \sigma)}$ are

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\} \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j, \sigma(i)\varepsilon_i \mid i \neq j\}$$

4. The roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$.
5. $\mathfrak{s}_{(\mathcal{P}, \sigma)} = \bigoplus_i \mathfrak{s}_{(\mathcal{P}, \sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i \cong gl_{|I_i|}$.
6. The Cartan subalgebra of $\mathfrak{s}_{\mathcal{P}}$ is spanned by $\{\sigma(j)h_j\}_{j \in I_i}$.
7. The roots of $\mathfrak{s}_{\mathcal{P}}$ are $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in I_i\}$.
8. $\mathfrak{a}_{(\mathcal{P}, \sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|I_i|} \sum_{j \in I_i} \sigma(j)h_j$.

If $\{\delta_1, \dots, \delta_k\}$ the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm \delta_i \pm \delta_j, \pm \delta_i \mid 1 \leq i \neq j \leq k\} \cup \{\pm 2\delta_i \mid |I_i| > 1\}$.
10. For $\nu \in \mathcal{R}$,

- (a) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm$ if $\nu = \pm \delta_i \pm \delta_j$,
- (b) $\mathfrak{g}^\nu \cong V_i^\pm$ if $\nu = \pm \delta_i$, and
- (c) $\mathfrak{g}^\nu \cong \Lambda^2 V_i^\pm$ if $\nu = \pm 2\delta_i$,

where V_i^+ and V_i^- respectively are the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ -module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ act trivially.

11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P}, \sigma}$ are in a bijection with the pairs (\mathcal{Q}, τ) such that the parts of \mathcal{Q} are the same as the parts of \mathcal{P} and $\sigma|_{I_i} = \pm \tau|_{I_i}$ for every part I_i or, equivalently, with linear orders on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$ compatible with multiplication by -1 .

In Type II:

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are

$$\begin{aligned} & \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j) \prec I_0\} \cup \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i \neq j \in I_0\} \\ & \quad \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j, \sigma(i)\varepsilon_i \mid i \neq j \notin I_0\} \cup \{\sigma(i)\varepsilon_i \pm \varepsilon_j \mid i \notin I_0, j \in I_0\} \end{aligned}$$

4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j) \prec I_0\} \cup \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i \neq j \in I_0\}.$$

5. $\mathfrak{s}_{(\mathcal{P},\sigma)} = \bigoplus_i \mathfrak{s}_{(\mathcal{P},\sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P},\sigma)}^0 \cong B_{|I_0|}$ and $\mathfrak{s}_{(\mathcal{P},\sigma)}^i \cong gl_{|I_i|}$ for $i > 0$.

6. The Cartan subalgebra of $\mathfrak{s}_{\mathcal{P}}^i$ is spanned by $\{h_j\}_{j \in I_0}$ for $i = 0$ and $\{\sigma(j)h_j\}_{j \in I_i}$ for $i > 0$.

7. The roots of $\mathfrak{s}_{\mathcal{P}}^i$ are $\{\pm\varepsilon_j \pm \varepsilon_l, \pm\varepsilon_j \mid j \neq l \in I_0\}$ for $i = 0$ and $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in I_i\}$ for $i > 0$.

8. $\mathfrak{a}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|I_i|} \sum_{j \in I_i} \sigma(j)h_j$.

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm\delta_i \pm \delta_j, \pm\delta_i \mid 1 \leq i \neq j \leq k\} \cup \{\pm 2\delta_i \mid |I_i| > 1\}$.

10. For $\nu \in \mathcal{R}$,

- (a) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm$ if $\nu = \pm\delta_i \pm \delta_j$,
- (b) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_0$ if $\nu = \pm\delta_i$, and
- (c) $\mathfrak{g}^\nu \cong \Lambda^2 V_i^\pm$ if $\nu = \pm 2\delta_i$

where V_i^+ and V_i^- denote the natural $\mathfrak{s}_{(\mathcal{P},\sigma)}^i$ -module and its dual respectively for $i > 0$, V_0 denotes the natural $\mathfrak{s}_{(\mathcal{P},\sigma)}^0$ -module, and all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially.

11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ are in a bijection with the pairs (Q, τ) such that the parts of Q are the same as the parts of \mathcal{P} and $\sigma|_{I_i} = \pm\tau|_{I_i}$ for every part I_i or, equivalently, with linear orders on the set $\{\pm\delta_1, \dots, \pm\delta_k\}$ compatible with multiplication by -1 .

$$\underline{\mathfrak{g} = C_n}$$

1. The roots of \mathfrak{g} are: $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}$.

2. Parabolic subalgebras of \mathfrak{g} are in one-to-one correspondence with:

Type I: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ and $\sigma: \{1, \dots, n\} \rightarrow \{\pm 1\}$ is a choice of signs.

Type II: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_0, I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ with largest element I_0 and $\sigma: \{1, \dots, n\} \setminus I_0 \rightarrow \{\pm 1\}$ is a choice of signs.

In Type I:

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\} \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j, 2\sigma(i)\varepsilon_i \mid i \neq j\}$$

4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$.

5. $\mathfrak{s}_{(\mathcal{P},\sigma)} = \bigoplus_i \mathfrak{s}_{(\mathcal{P},\sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P},\sigma)}^i \cong gl_{|I_i|}$.

6. The Cartan subalgebra of $\mathfrak{s}_{\mathcal{P}}^i$ is spanned by $\{\sigma(j)h_j\}_{j \in I_i}$.

7. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}^i$ are $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in I_i\}$.

8. $\mathfrak{a}_{(\mathcal{P}, \sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|\mathbb{I}_i|} \sum_{j \in \mathbb{I}_i} \sigma(j) h_j$.

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm \delta_i \pm \delta_j, \pm 2\delta_i \mid 1 \leq i \neq j \leq k\}$.

10. For $\nu \in \mathcal{R}$,

- (a) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm$ if $\nu = \pm \delta_i \pm \delta_j$.
- (b) $\mathfrak{g}^\nu \cong \text{Sym}^2 V_i^\pm$ if for $\nu = \pm 2\delta_i$.

where V_i^+ and V_i^- are the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ -module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ act trivially.

11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P}, \sigma}$ are in a bijection with the pairs (Q, τ) such that the parts of Q are the same as the parts of \mathcal{P} and $\sigma|_{\mathbb{I}_i} = \pm \tau|_{\mathbb{I}_i}$ for every part \mathbb{I}_i or, equivalently, with linear orders on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$ compatible with multiplication by -1 .

In Type II:

3. The roots of $\mathfrak{p}_{(\mathcal{P}, \sigma)}$ are

$$\begin{aligned} & \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j) \prec \mathbb{I}_0\} \cup \{\pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i \neq j \in \mathbb{I}_0\} \cup \\ & \quad \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j, \sigma(i)2\varepsilon_i \mid i \neq j \notin \mathbb{I}_0\} \cup \{\sigma(i)\varepsilon_i \pm \varepsilon_j \mid i \notin \mathbb{I}_0, j \in \mathbb{I}_0\} \end{aligned}$$

4. The roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ are

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j) \prec \mathbb{I}_0\} \cup \{\pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i \neq j \in \mathbb{I}_0\}.$$

5. $\mathfrak{s}_{(\mathcal{P}, \sigma)} = \bigoplus_{i=0}^k \mathfrak{s}_{(\mathcal{P}, \sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P}, \sigma)}^0 \cong C_{|\mathbb{I}_0|}$ and $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i \cong gl_{|\mathbb{I}_i|}$ for $i > 0$.

6. The Cartan subalgebra of $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ is spanned by $\{h_j\}_{j \in \mathbb{I}_0}$ for $i = 0$ and $\{\sigma(j)h_j\}_{j \in \mathbb{I}_i}$ for $i > 0$.

7. The roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ are $\{\pm \varepsilon_j \pm \varepsilon_l, \pm 2\varepsilon_j \mid j \neq l \in \mathbb{I}_0\}$ for $i = 0$ and $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in \mathbb{I}_i\}$ for $i > 0$.

8. $\mathfrak{a}_{(\mathcal{P}, \sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|\mathbb{I}_i|} \sum_{j \in \mathbb{I}_i} \sigma(j) h_j$

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm \delta_i \pm \delta_j, \pm \delta_i, \pm 2\delta_i \mid 1 \leq i \neq j \leq k\}$.

10. For $\nu \in \mathcal{R}$,

- (a) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm$ if $\nu = \pm \delta_i \pm \delta_j$,
- (b) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_0$ if $\nu = \pm \delta_i$,
- (c) \mathfrak{g}^ν is isomorphic to $\text{Sym}^2 V_i^\pm$ if $\nu = \pm 2\delta_i$,

where V_i^+ and V_i^- denote the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ -module and its dual for $i > 0$, V_0 is the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^0$ -module, and where all other factors of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ act trivially.

11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P}, \sigma}$ are in a bijection with the pairs (Q, τ) such that the parts of Q are the same as the parts of \mathcal{P} and $\sigma|_{\mathbb{I}_i} = \pm \tau|_{\mathbb{I}_i}$ for every part \mathbb{I}_i or, equivalently, with linear orders on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$ compatible with multiplication by -1 .

$\mathfrak{g} = D_n$

1. The roots of \mathfrak{g} are: $\Delta = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq n\}$.

2. Parabolic subalgebras of \mathfrak{g} are determined by:

Type I: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ and $\sigma : \{1, \dots, n\} \rightarrow \{\pm 1\}$ is a choice of signs.

Two pairs (\mathcal{P}', σ') and $(\mathcal{P}'', \sigma'')$ determine the same parabolic subalgebra if and only if \mathcal{P}' and \mathcal{P}'' are the same ordered partitions whose maximal part I_0 contains one element and σ' and σ'' coincide on $\{1, \dots, n\} \setminus I_0$.

Type II: pairs (\mathcal{P}, σ) , where $\mathcal{P} = (I_0, I_1, \dots, I_k)$ is a linearly ordered partition of $\{1, \dots, n\}$ with largest element I_0 such that $|I_0| \geq 2$ and $\sigma : \{1, \dots, n\} \setminus I_0 \rightarrow \{\pm 1\}$ is a choice of signs.

In Type I:

3. The roots of $\mathfrak{p}_{(\mathcal{P}, \sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\} \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j \mid i \neq j\}$
4. The roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$.
5. $\mathfrak{s}_{(\mathcal{P}, \sigma)} = \bigoplus_i \mathfrak{s}_{(\mathcal{P}, \sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i \cong gl_{|I_i|}$.
6. The Cartan subalgebra of $\mathfrak{s}_{\mathcal{P}}^i$ is spanned by $\{\sigma(j)h_j\}_{j \in I_i}$
7. The roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ are $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in I_i\}$.
8. $\mathfrak{a}_{(\mathcal{P}, \sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|I_i|} \sum_{j \in I_i} \sigma(j)h_j$.

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm \delta_i \pm \delta_j \mid 1 \leq i \neq j \leq k\} \cup \{2\delta_i \mid |I_i| > 1\}$.
10. For $\nu \in \mathcal{R}$,
 - (a) $\mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm$ if $\nu = \pm \delta_i \pm \delta_j$.
 - (b) $\mathfrak{g}^\nu \cong \Lambda^2 V_i^\pm$ if $\nu = \pm 2\delta_i$.

where V_i^+ and V_i^- are the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ -module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ act trivially.

11. Every parabolic subalgebra of \mathfrak{g} whose reductive part is $\mathfrak{s}_{\mathcal{P}, \sigma}$ corresponds to a pair (\mathcal{Q}, τ) such that the parts of \mathcal{Q} are the same as the parts of \mathcal{P} and $\sigma|_{I_i} = \pm \tau|_{I_i}$ for every part I_i or, equivalently, to a linear order on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$ compatible with multiplication by -1 . Note that this correspondence is not bijective since two different linear orders may determine the same parabolic subalgebra.

In Type II:

3. Roots of $\mathfrak{p}_{(\mathcal{P}, \sigma)} =$

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j) \prec I_0\} \cup \{\pm \varepsilon_i \pm \varepsilon_j \mid i \neq j \in I_0\}$$

$$\cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j \mid i \neq j \notin I_0\} \cup \{\sigma(i)\varepsilon_i \pm \varepsilon_j \mid i \notin I_0, j \in I_0\}$$
4. Roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)} = \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \mid i \neq j, \mathcal{P}(i) = \mathcal{P}(j) \prec I_0\} \cup \{\pm \varepsilon_i \pm \varepsilon_j \mid i \neq j \in I_0\}$.
5. $\mathfrak{s}_{(\mathcal{P}, \sigma)} = \bigoplus_{i=0}^k \mathfrak{s}_{(\mathcal{P}, \sigma)}^i$, where $\mathfrak{s}_{(\mathcal{P}, \sigma)}^0 \cong D_{|I_0|}$ and $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i \cong gl_{|I_i|}$ for $i > 0$.
6. Cartan subalgebra of $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ is spanned by $\{h_j\}_{j \in I_0}$ for $i = 0$ and by $\{\sigma(j)h_j\}_{j \in I_i}$ for $i > 0$;

7. roots of $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ are $\{\pm \varepsilon_j \pm \varepsilon_l \mid j \neq l \in I_0\}$ for $i = 0$ and $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \mid j \neq l \in I_i\}$ for $i > 0$.

8. $\mathfrak{a}_{(\mathcal{P}, \sigma)}$ has a basis $\{t_1, \dots, t_k\}$ with $t_i = \frac{1}{|I_i|} \sum_{j \in I_i} \sigma(j)h_j$.

If $\{\delta_1, \dots, \delta_k\}$ is the basis of \mathfrak{a}^* dual to $\{t_1, \dots, t_k\}$ then

9. $\mathcal{R} = \{\pm \delta_i \pm \delta_j, \pm \delta_i \mid 1 \leq i \neq j \leq k\} \cup \{\pm 2\delta_i \mid |I_k| > 1\}$.

10. For $\nu \in \mathcal{R}$,

$$(a) \mathfrak{g}^\nu \cong V_i^\pm \otimes V_j^\pm \text{ if } \nu = \pm \delta_i \pm \delta_j,$$

$$(b) \mathfrak{g}^\nu \cong V_i^\pm \otimes V_0 \text{ if } \nu = \pm \delta_i,$$

$$(c) \mathfrak{g}^\nu \cong \Lambda^2 V_i^\pm \text{ if } \nu = \pm 2\delta_i,$$

where V_i^+ and V_i^- denote the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^i$ -module and its dual for $i > 0$, V_0 is the natural $\mathfrak{s}_{(\mathcal{P}, \sigma)}^0$ -module, and where all other factors of $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ act trivially.

11. The parabolic subalgebras of \mathfrak{g} whose reductive part is $\mathfrak{s}_{(\mathcal{P}, \sigma)}$ are in a bijection with the pairs (Q, τ) such that the parts of Q are the same as the parts of \mathcal{P} and $\sigma|_{I_i} = \pm \tau|_{I_i}$ for every part I_i or, equivalently, with linear orders on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$.

Proof of Theorem A.1. The idea is simple: using \mathcal{S} we will define a partial order on the set $\{\delta_1, \dots, \delta_k\}$ (respectively on $\{\pm \delta_1, \dots, \pm \delta_k\}$) and using the fact that $\dim(\text{Sym}^*(\mathcal{M}))^\mathfrak{s} = 1$ we will prove that this order can be extended to a linear order (respectively, to a linear order compatible with multiplication by -1). We show this case-by-case.

Case 1: $\mathfrak{g} = gl_n$.

Set

$$(A.2) \quad \delta_i \prec \delta_j \quad \text{if} \quad \nu = \delta_i - \delta_j \in \mathcal{S}.$$

We claim that \prec can be extended to a linear order on the set $\{\delta_1, \dots, \delta_k\}$. Assume not. Then there is a sequence $\nu_1 = \delta_{i_1} - \delta_{i_2}, \nu_2 = \delta_{i_2} - \delta_{i_3}, \dots, \nu_l = \delta_{i_l} - \delta_{i_1}$ of elements of \mathcal{S} . We may assume that this sequence is the shortest among all such sequences, and hence that the \mathfrak{s} -modules $\mathfrak{g}^{\nu_1}, \dots, \mathfrak{g}^{\nu_l}$ are distinct submodules of \mathcal{M} . As a consequence, $\mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_l}$ is a submodule of $\text{Sym}^*(\mathcal{M})$. On the other hand,

$$(A.3) \quad \mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_l} \cong (V_{i_1} \otimes V_{i_2}^*) \otimes \dots \otimes (V_{i_l} \otimes V_{i_1}^*) \cong (V_{i_1} \otimes V_{i_1}^*) \otimes \dots \otimes (V_{i_l} \otimes V_{i_l}^*),$$

where the lower index of a module shows which component of $\mathfrak{s}_\mathcal{P}$ acts non-trivially on it. The right hand side module in (A.3) clearly contains the trivial $\mathfrak{s}_\mathcal{P}$ -module which contradicts the assumption that $\dim(\text{Sym}^*(\mathcal{M}))^\mathfrak{s} = 1$. This contradiction shows that the order (A.2) can be extended to a linear order on the set $\{\delta_1, \dots, \delta_k\}$, which completes the proof in this case.

Case 2: $\mathfrak{g} = B_n$, Type I.

Set

$$(A.4) \quad \begin{aligned} s_i \delta_i \prec s_j \delta_j \text{ for } i \neq j &\quad \text{if} \quad \nu = s_i \delta_i - s_j \delta_j \in \mathcal{S} \\ s_i \delta_i \prec -s_i \delta_i &\quad \text{if} \quad \nu = s_i \delta_i \in \mathcal{S}, \end{aligned}$$

where $s_i, s_j = \pm 1$. We claim that \prec can be extended to a linear order on the set $\{\pm \delta_1, \dots, \pm \delta_k\}$ compatible with multiplication by -1 . Assume not. Then there is a cycle

$$(A.5) \quad s_1\delta_{i_1} \prec s_2\delta_{i_2} \prec \dots \prec s_l\delta_{i_l} \prec s_1\delta_{i_1}$$

of elements of $\{\pm\delta_1, \dots, \pm\delta_k\}$. We may assume that we have chosen this cycle to be of minimal length. It gives rise to a sequence $\nu_1, \dots, \nu_l \in \mathcal{S}$ induced from (A.4). More precisely,

$$\nu_j = \begin{cases} s_j\delta_{i_j} - s_{j+1}\delta_{i_{j+1}} & \text{if } \delta_{i_j} \neq \delta_{i_{j+1}} \\ s_j\delta_{i_j} & \text{if } \delta_{i_j} = \delta_{i_{j+1}}, \end{cases}$$

where $s_{l+1} = s_1$ and $\delta_{i_{l+1}} = \delta_{i_1}$. Notice that the minimality of the cycle (A.5) implies that $\delta_{i_j} = \delta_{i_{j+1}}$ for at most two values of j .

If $\delta_{i_j} \neq \delta_{i_{j+1}}$ for every j , then, as in Case 1, $\mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_l}$ is a submodule of $\text{Sym}^\cdot(\mathcal{M})$ and

$$\mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_l} \cong (V_{i_1}^{s_1} \otimes V_{i_2}^{-s_2}) \otimes \dots \otimes (V_{i_l}^{s_l} \otimes V_{i_1}^{-s_1}) \cong (V_{i_1}^{s_1} \otimes V_{i_1}^{-s_1}) \otimes \dots \otimes (V_{i_l}^{s_l} \otimes V_{i_l}^{-s_l})$$

contains the trivial \mathfrak{s}_P -module which contradicts the assumption that $\dim(\text{Sym}^\cdot(\mathcal{M}))^\mathfrak{s} = 1$.

If $\delta_{i_1} = \delta_{i_l}$ (which we may assume in case there is only one j with $\delta_{i_j} = \delta_{i_{j+1}}$), then $s_l = -s_1$. Furthermore, $\mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_{l-1}} \otimes \text{Sym}^2 \mathfrak{g}^{\nu_l}$ is a submodule of $\text{Sym}^\cdot(\mathcal{M})$ and

$$\begin{aligned} \mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_{l-1}} \otimes \text{Sym}^2 \mathfrak{g}^{\nu_l} &\cong (V_{i_1}^{s_1} \otimes V_{i_2}^{-s_2}) \otimes \dots \otimes (V_{i_{l-1}}^{s_{l-1}} \otimes V_{i_1}^{s_1}) \otimes (\text{Sym}^2 V_{i_1}^{-s_1}) \\ &\cong ((V_{i_1}^{s_1})^{\otimes 2} \otimes \text{Sym}^2 V_{i_1}^{-s_1}) \otimes (V_{i_2}^{s_2} \otimes V_{i_2}^{-s_2}) \otimes \dots \otimes (V_{i_{l-1}}^{s_{l-1}} \otimes V_{i_{l-1}}^{-s_{l-1}}) \end{aligned}$$

contains the trivial \mathfrak{s}_P -module which contradicts the assumption that $\dim(\text{Sym}^\cdot(\mathcal{M}))^\mathfrak{s} = 1$.

Finally, if $\delta_{i_j} = \delta_{i_{j+1}}$ for two values of j we proceed as above. The only difference is that we will obtain a term analogous to $(V_{i_1}^{s_1})^{\otimes 2} \otimes \text{Sym}^2 V_{i_1}^{-s_1}$ twice. The proof is complete in this case.

Case 3: $\mathfrak{g} = B_n, C_n$ or D_n , Type II.

This case is analogous to Case 2. The only difference is that if $\nu = s_i\delta_i$ then $\mathfrak{g}^\nu \cong V_i^{s_i} \otimes V_0$ and consequently

$$\text{Sym}^2 \mathfrak{g}^\nu \cong \text{Sym}^2 V_i^{s_i} \otimes \text{Sym}^2 V_0 \oplus \Lambda^2 V_i^{s_i} \otimes \Lambda^2 V_0.$$

The observation that the trivial $\mathfrak{s}_{(P,\sigma)}^0$ -module is contained in $(\text{Sym}^2 V_0)^\mathfrak{s}$ for $\mathfrak{g} = B_n, D_n$ and in $\Lambda^2 V_0$ for $\mathfrak{g} = C_n$ concludes the proof.

Case 4: $\mathfrak{g} = C_n$ or D_n , Type I.

This case is again analogous to Case 2. The only difference is that we need to modify the second line of formula (A.4) to read

$$(A.6) \quad s_i\delta_i \prec -s_i\delta_i \quad \text{if} \quad \nu = s_i2\delta_i \in \mathcal{S}.$$

Consequently, for ν as in (A.6) we have $\mathfrak{g}^\nu \cong \text{Sym}^2 V_i^{s_i}$ for $\mathfrak{g} = C_n$ and $\mathfrak{g}^\nu \cong \Lambda^2 V_i^{s_i}$ for $\mathfrak{g} = D_n$ and proceeding as in Case 2 we find a non-trivial invariant in the module $\mathfrak{g}^{\nu_1} \otimes \dots \otimes \mathfrak{g}^{\nu_l} \subset (\text{Sym}^\cdot \mathcal{M})^\mathfrak{s}$. \square

Exceptional Lie algebras. We first note that Theorem A.1 holds for $\mathfrak{g} = G_2$. Indeed, if \mathfrak{s} is a proper subalgebra of \mathfrak{g} which is not equal to \mathfrak{t} , then all elements of \mathcal{R} are proportional

and there is nothing to prove. If $\mathfrak{s} = \mathfrak{t}$, then the spaces \mathfrak{g}^ν are just the root spaces of \mathfrak{g} which are one dimensional and again the statement is clear.

The proposition, however, fails to hold for the remaining exceptional Lie algebras. Here is a uniform example. Let $\mathfrak{g} = F_4, E_6, E_7$, or E_8 . Denote the rank of \mathfrak{g} by n . Then \mathfrak{g} has a reductive subalgebra $\mathfrak{m} \oplus \mathfrak{c}$ of rank n such that $\mathfrak{m} \cong A_1$ and $\mathfrak{c} \cong C_3, A_5, D_6$, or E_7 respectively. As an \mathfrak{m} -module \mathfrak{g} decomposes as

$$(A.7) \quad \mathfrak{g} = (\text{Ad}_{\mathfrak{m}} \otimes \text{tr}_{\mathfrak{c}}) \oplus (\text{tr}_{\mathfrak{m}} \otimes \text{Ad}_{\mathfrak{c}}) \oplus (V \otimes U),$$

where $\text{Ad}_{\mathfrak{m}}$ and $\text{Ad}_{\mathfrak{c}}$ are the adjoint modules of \mathfrak{m} and \mathfrak{c} respectively; $\text{tr}_{\mathfrak{m}}$ and $\text{tr}_{\mathfrak{c}}$ —the respective trivial modules; V is the natural $\mathfrak{m} \cong A_1$ -module; and the weights of U belong to a single \mathcal{W} -orbit.

Set $\mathfrak{s} = \mathfrak{m} + \mathfrak{t}$. From the construction of \mathfrak{s} we conclude that $\mathfrak{a} = \mathfrak{t}_{\mathfrak{c}}$, the Cartan subalgebra of \mathfrak{c} . Furthermore, (A.7) implies $\mathcal{R} = \Delta_{\mathfrak{c}} \cup \text{supp } U$, where $\text{supp } U$ denotes the set of weights of U and, for $\nu \in \mathcal{R}$ the $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{t}_{\mathfrak{c}}$ -module \mathfrak{g}^ν is given by

$$\mathfrak{g}^\nu \cong \begin{cases} \text{tr}_{\mathfrak{m}} \otimes \nu & \text{if } \nu \in \Delta_{\mathfrak{c}} \\ V \otimes \nu & \text{if } \nu \in \text{supp } U. \end{cases}$$

Pick $\nu_0 \in \text{supp } U$ and write $\nu_0 = q_1\nu_1 + \dots + q_r\nu_r$ with $q_i \in \mathbb{Q}_+$, $\nu_i \in \Delta_{\mathfrak{c}}^+$. Set $\mathcal{S} = \{-\nu_0, \nu_1, \dots, \nu_r\}$ and $\mathcal{M} = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^\nu$. It is clear from the construction that $\dim \text{Sym}(\mathcal{M})^{\mathfrak{s}} = 1$ but no parabolic subalgebra \mathfrak{p} as in Theorem A.1 exists.

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